

Numerical optimization and optimal control for dynamical systems [146442]

Summary of Results

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Preface

This volume collects, chapter by chapter, the essential results of the course *Numerical optimization and optimal control for dynamical systems*: problem statements, conditions, procedures and typical traps, without proofs and with almost no examples. It is meant to be kept at hand while solving exercises and written exams. The full treatment, with derivations and examples, is in the course notes, whose parts and chapter names this book mirrors; the exercises are in the companion volume.

How to read it. Each topic is organized as **Problem** (what is minimized and under which constraints), **Conditions** (the system to solve, one label per equation), **Recipe** (the steps to follow in an exercise) and **Watch out** (the usual mistakes). Each chapter ends with a one-page summary.

Colors. The same color always plays the same role:

color	role
blue	objective, running cost L
green	endpoint cost Φ
royal blue	dynamics, differential equations
brown	equality and integral constraints, algebraic equations
orchid	inequality constraints, complementarity, control bounds
orange	boundary conditions, variations δx
crimson	labels of equations and of derivation steps

1. Constrained Minimization

Everything needed to set up and solve a constrained minimization exercise: Lagrange multipliers, definiteness tests, KKT conditions, second-order tests. Colors: objective, equality constraints, inequality constraints and complementarity, labels.

1.1 Lagrange Multipliers

Problem. $f \in C^2(\mathbb{R}^n, \mathbb{R})$, $\boldsymbol{h} \in C^2(\mathbb{R}^n, \mathbb{R}^m)$, $m \leq n$. Gradients of scalar functions are row vectors; $\boldsymbol{J} = D\boldsymbol{h}$ has the rows ∇h_i.

$$\begin{aligned} &\text{minimize:} && f(\boldsymbol{x}) \\ &\text{subject to:} && h_i(\boldsymbol{x}) = 0, \quad i = 1, \dots, m \end{aligned}$$

Lagrangian.

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}) = f(\boldsymbol{x}) - \boldsymbol{\lambda}^T \boldsymbol{h}(\boldsymbol{x})$$

Conditions. At a local minimizer with $\operatorname{rank} \boldsymbol{J}(\boldsymbol{x}^\star) = m$ (LICQ, linear independence constraint qualification) the first-order, feasibility and **Second-Order Necessary Conditions (SONC)** hold:

$$\begin{cases} \nabla f(\boldsymbol{x}^\star) - (\boldsymbol{\lambda}^\star)^T \boldsymbol{J}(\boldsymbol{x}^\star) = \boldsymbol{0} & \text{first order} \\ \boldsymbol{h}(\boldsymbol{x}^\star) = \boldsymbol{0} & \text{feasibility} \\ \boldsymbol{z}^T \nabla^2_{\boldsymbol{x}\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)\, \boldsymbol{z} \geq 0, \quad \forall\, \boldsymbol{z} \in \ker \boldsymbol{J}(\boldsymbol{x}^\star) & \text{SONC} \end{cases}$$

Sufficient. If $\boldsymbol{h}(\boldsymbol{x}^\star) = \boldsymbol{0}$, $\nabla_{\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star) = \boldsymbol{0}$ and the **Second-Order Sufficient Condition (SOSC)** holds,

$$\boldsymbol{z}^T \nabla^2_{\boldsymbol{x}\boldsymbol{x}} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)\, \boldsymbol{z} > 0, \quad \forall\, \boldsymbol{z} \in \ker \boldsymbol{J}(\boldsymbol{x}^\star) \setminus \{\boldsymbol{0}\} \quad \text{SOSC}$$

then $\boldsymbol{x}^\star$ is a strict local minimizer (no rank condition needed).

Recipe.

1. Form $\mathcal{L}$; solve $\nabla_{\boldsymbol{x}} \mathcal{L} = \boldsymbol{0}$, $\boldsymbol{h} = \boldsymbol{0}$ ($n + m$ equations in $n + m$ unknowns $\boldsymbol{x}, \boldsymbol{\lambda}$).

2. At each candidate check rank $\boldsymbol{J} = m$.
3. Also find the feasible points with rank $\boldsymbol{J} < m$: multipliers need not exist there, they may be minimizers, and they are studied directly.
4. Tangent space $\ker \boldsymbol{J}$ ($\dim = n - m$ under LICQ): basis $\boldsymbol{K} \in \mathbb{R}^{n\times r}$, $r = n - \operatorname{rank} \boldsymbol{J}$, $\boldsymbol{J}\boldsymbol{K} = \boldsymbol{0}$.
5. Reduced Hessian $\boldsymbol{H}_R = \boldsymbol{K}^T \nabla^2_{\boldsymbol{xx}} \mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)\boldsymbol{K} \in \mathbb{R}^{r\times r}$.
6. Classify with the table below.
7. If the feasible set is compact, compare f at all candidates and at the rank-deficient feasible points: global minimum and maximum.

$\boldsymbol{H}_R$ at $(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star)$	verdict
$\succ 0$	strict local minimum
$\prec 0$	strict local maximum
indefinite	neither minimum nor maximum (saddle); needs rank $\boldsymbol{J} = m$
semidefinite, singular	inconclusive (SONC holds, SOSC does not)
$r = 0$	test vacuous: isolated feasible point, strict local minimum (and maximum)

Watch out.

- The second-order object is the Hessian of $\mathcal{L}$, not of f: a curved constraint contributes $-\sum_i \lambda_i \nabla^2 h_i$. For linear constraints $\nabla^2 \mathcal{L} = \nabla^2 f$.
- Definiteness of $\boldsymbol{H}_R$ does not depend on the choice of the basis $\boldsymbol{K}$.
- Maximize f: minimize $-f$ (then $\boldsymbol{\lambda}$ and $\nabla^2 \mathcal{L}$ change sign), or read the $\prec 0$ row.
- No constraints ($m = 0$): $\boldsymbol{K} = \boldsymbol{I}$, $\boldsymbol{H}_R = \nabla^2 f$. Constraints solvable by substitution: eliminate them and use the unconstrained tests.

1.2 Sylvester's Criterion

Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be **symmetric**. If it is not, only $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$ matters ($\boldsymbol{z}^T\boldsymbol{A}\boldsymbol{z} = \boldsymbol{z}^T\boldsymbol{S}\boldsymbol{z}$): apply every test below to $\boldsymbol{S}$. $\Delta_k = \det \boldsymbol{A}_{1:k,1:k}$ is the **leading** principal minor of order k. A **principal** minor of order k is $\det \boldsymbol{A}_{I,I}$ for any index set I, $|I| = k$ (there are $\binom{n}{k}$ of them, $2^n - 1$ in total).

matrix	test
$\boldsymbol{A} \succ 0$	$\Delta_k > 0$, $k = 1, \dots, n$
$\boldsymbol{A} \succeq 0$	**every** principal minor ≥ 0 (all orders, all index sets I)
equivalent	eigenvalues: $\succ 0 \iff \lambda_i(\boldsymbol{A}) > 0$; $\succeq 0 \iff \lambda_i(\boldsymbol{A}) \geq 0$ for all i

For $\boldsymbol{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$: $\boldsymbol{A} \succ 0 \iff a > 0,\ ac - b^2 > 0$; $\boldsymbol{A} \succeq 0 \iff a \geq 0,\ c \geq 0,\ ac - b^2 \geq 0$.

Positive semidefinite in practice.

1. Quick necessary checks: $a_{ii} \geq 0$ and $a_{ij}^2 \leq a_{ii}a_{jj}$ for all i, j. In particular $a_{ii} = 0$ forces row and column i to be zero: delete them and test the smaller matrix.
2. If $\Delta_1, \dots, \Delta_{n-1} > 0$, then $\boldsymbol{A} \succeq 0 \iff \det \boldsymbol{A} \geq 0$: only the last leading minor may vanish.

3. Otherwise check all principal minors. For $n = 3$ they are seven: $a_{11}, a_{22}, a_{33} \geq 0$; $a_{11}a_{22} - a_{12}^2,\ a_{11}a_{33} - a_{13}^2,\ a_{22}a_{33} - a_{23}^2 \geq 0$; $\det \boldsymbol{A} \geq 0$.
4. Without minors (symmetric elimination, $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{D}\boldsymbol{L}^T$). Bring any diagonal entry to position 1 by a symmetric permutation and write $\boldsymbol{A} = \begin{bmatrix} a & \boldsymbol{b}^T \\ \boldsymbol{b} & \boldsymbol{C} \end{bmatrix}$:
   - $a < 0$: $\boldsymbol{A} \not\succeq 0$;
   - $a = 0$: need $\boldsymbol{b} = \boldsymbol{0}$, then continue with $\boldsymbol{C}$;
   - $a > 0$: continue with the Schur complement $\boldsymbol{C} - \boldsymbol{b}\boldsymbol{b}^T/a$.

   $\boldsymbol{A} \succeq 0$ iff no negative pivot appears; $\boldsymbol{A} \succ 0$ iff all n pivots are > 0.

Recipe. Apply the test to $\boldsymbol{H}_R = \boldsymbol{K}^T \nabla^2_{xx}\mathcal{L}\, \boldsymbol{K}$ (size $r \times r$), not to $\nabla^2_{xx}\mathcal{L}$.

Watch out.

- Leading minors ≥ 0 do **not** imply $\succeq 0$: $\operatorname{diag}(0, -1)$ has $\Delta_1 = \Delta_2 = 0$ but is not positive semidefinite ($a_{22} = -1 < 0$). Also $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ has $\Delta_1 = 1$, $\Delta_2 = \Delta_3 = 0$, yet the principal minor on $I = \{1, 3\}$ is -1. For semidefiniteness use all principal minors.
- $\succ 0$ needs only the leading minors, strictly positive.
- Positive definiteness on the whole $\mathbb{R}^n$ is not required: only on $\ker \boldsymbol{J}$.
- Leading minors of a non-symmetric matrix prove nothing: $\begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix}$ has $\Delta_1 = 1$, $\Delta_2 = 4$, yet $\boldsymbol{z}^T\boldsymbol{A}\boldsymbol{z} = (z_1 + z_2)^2$ is only semidefinite.

1.3 Inequality Constraints and KKT Conditions

Problem. f, h_i, g_ℓ of class C^1 (C^2 for second order). Inequalities are written $g_\ell \geq 0$.

$$\begin{aligned} \text{minimize:} \quad & f(\boldsymbol{x}) \\ \text{subject to:} \quad & h_i(\boldsymbol{x}) = 0, \quad i = 1, \dots, m \\ & g_\ell(\boldsymbol{x}) \geq 0, \quad \ell = 1, \dots, p \end{aligned}$$

Lagrangian (sign convention):

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{x}) - \sum_{i=1}^{m} \lambda_i h_i(\boldsymbol{x}) - \sum_{\ell=1}^{p} \mu_\ell g_\ell(\boldsymbol{x})$$

KKT conditions. At a local minimizer where a constraint qualification (LICQ; MFCQ, Mangasarian–Fromovitz constraint qualification; …) holds there are $\boldsymbol{\lambda}^\star$, $\boldsymbol{\mu}^\star$ with

$$\begin{cases} \nabla f(\boldsymbol{x}^\star) - \sum_i \lambda_i^\star \nabla h_i(\boldsymbol{x}^\star) - \sum_\ell \mu_\ell^\star \nabla g_\ell(\boldsymbol{x}^\star) = \boldsymbol{0} & \text{stationarity} \\ h_i(\boldsymbol{x}^\star) = 0 & \text{primal feasibility} \\ g_\ell(\boldsymbol{x}^\star) \geq 0 & \text{primal feasibility} \\ \mu_\ell^\star \geq 0 & \text{dual feasibility} \\ \mu_\ell^\star\, g_\ell(\boldsymbol{x}^\star) = 0 & \text{complementarity} \end{cases}$$

Hence $g_\ell(\boldsymbol{x}^\star) > 0 \Rightarrow \mu_\ell^\star = 0$. The multipliers λ_i are unrestricted in sign. Under LICQ the multipliers are unique.

Active set. $\mathcal{F} = \{\boldsymbol{x} : \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{0},\ \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{0}\}$ is the feasible set, $\mathcal{A}(\boldsymbol{x}^\star) = \{\ell : g_\ell(\boldsymbol{x}^\star) = 0\}$; inactive constraints do not matter locally.

Recipe (KKT by cases).

1. Bring the problem to the form above: $g \leq 0$ becomes $-g \geq 0$; maximize f becomes minimize $-f$.
2. Form $\mathcal{L}$ with $-\mu_\ell g_\ell$ and $\mu_\ell \geq 0$.
3. For each guess of the active set: set $g_\ell = 0$ for $\ell \in \mathcal{A}$ and $\mu_\ell = 0$ for $\ell \notin \mathcal{A}$; solve stationarity and $\boldsymbol{h} = \boldsymbol{0}$.
4. Discard solutions with some $g_\ell < 0$ or some $\mu_\ell < 0$.
5. At each survivor check the CQ (below).
6. Also find the feasible points where the CQ fails ($\boldsymbol{h} = \boldsymbol{0}$, $g_\ell = 0$ for $\ell \in \mathcal{A}$, dependent gradients): they may be minimizers without multipliers, so examine them directly (feasible curves, bounds).
7. Second-order test on the critical cone (below).
8. If the feasible set is compact, compare f at the KKT points and at the CQ-failing points: the global minimum is among them.

Watch out.

- Sign convention: with $g_\ell \geq 0$ and $\mathcal{L} = f - \sum \lambda_i h_i - \sum \mu_\ell g_\ell$ the condition is $\mu_\ell \geq 0$. With another sign convention the signs flip consistently.
- No CQ, no multipliers guaranteed: e.g. two active constraints with $\nabla g_1 = -\nabla g_2 \neq \boldsymbol{0}$ (no $\boldsymbol{d}$ with $\nabla g_1 \boldsymbol{d} > 0$ and $\nabla g_2 \boldsymbol{d} > 0$). Decide by a feasible curve or a bound.
- Squared slacks $g_\ell = \varepsilon_\ell^2$ give an equality problem, but first order yields only $\mu_\ell \varepsilon_\ell = 0$: impose $\mu_\ell \geq 0$ separately (it comes from second order in ε_ℓ); at active constraints ($\varepsilon_\ell = 0$) equality regularity may fail.

1.3.1 Constraint qualifications

$\boldsymbol{x}^\star$ feasible.

CQ	test
LICQ	the gradients $\nabla h_i(\boldsymbol{x}^\star)$ and $\nabla g_\ell(\boldsymbol{x}^\star)$, $\ell \in \mathcal{A}$, are linearly independent: the stacked matrix has rank $m + \lvert\mathcal{A}\rvert$
MFCQ	$\nabla h_i(\boldsymbol{x}^\star)$ linearly independent **and** $\exists\, \boldsymbol{d}$: $\nabla h_i \boldsymbol{d} = 0$ for all i, $\nabla g_\ell \boldsymbol{d} > 0$ for $\ell \in \mathcal{A}$ (the active ∇g_ℓ may be linearly dependent). Equivalently: $\sum_{\ell \in \mathcal{A}} \alpha_\ell \nabla g_\ell + \sum_i \beta_i \nabla h_i = \boldsymbol{0}$ with $\alpha_\ell \geq 0$ forces all $\alpha_\ell = 0$, $\beta_i = 0$

LICQ ⇒ MFCQ ⇒ ACQ ⇒ GCQ (defined in the optional table below), no reverse implication; GCQ already gives KKT necessity. Linear constraints satisfy ACQ automatically.

Fritz–John (fallback when no CQ holds). A local minimizer has $\mu_0 \geq 0$, $\boldsymbol{\lambda}, \boldsymbol{\mu} \geq \boldsymbol{0}$, not all zero, with

$$\begin{cases} \mu_0 \nabla f - \displaystyle\sum_i \lambda_i \nabla h_i - \sum_\ell \mu_\ell \nabla g_\ell = \boldsymbol{0} & \text{stationarity} \\ \mu_\ell\, g_\ell = 0 & \text{complementarity} \end{cases}$$

If $\mu_0 > 0$, normalize $\mu_0 = 1$ and get KKT; $\mu_0 = 0$ is the abnormal case (only constraint gradients appear, no information on f). MFCQ excludes it.

1.3.2 Geometry of the feasible set *

Optional. $\mathcal{K}^\circ = \{\boldsymbol{y} : \boldsymbol{y}^T\boldsymbol{d} \le 0 \ \forall \boldsymbol{d} \in \mathcal{K}\}$ is the polar of a cone $\mathcal{K}$.

object	definition
tangent cone T	limits of $(\boldsymbol{x}_j - \boldsymbol{x}^\star)/t_j$, $\boldsymbol{x}_j \in \mathcal{F}$, $\boldsymbol{x}_j \to \boldsymbol{x}^\star$, $t_j \downarrow 0$; minimizer: $\nabla f(\boldsymbol{x}^\star)\boldsymbol{d} \ge 0 \ \forall \boldsymbol{d} \in T$
linearized cone $\mathcal{F}_L$	$\{\boldsymbol{d} : \nabla h_i \boldsymbol{d} = 0, \ \nabla g_\ell \boldsymbol{d} \ge 0, \ \ell \in \mathcal{A}\}$; always $T \subseteq \mathcal{F}_L$
ACQ / GCQ	$T = \mathcal{F}_L$ / $T^\circ = \mathcal{F}_L^\circ$

$T \ne \mathcal{F}_L$ is possible: $g = -x_1^2 \ge 0$ at the origin has $\nabla g = \boldsymbol{0}$, so $\mathcal{F}_L = \mathbb{R}^2$ while T is the vertical line.

1.3.3 Second-order conditions

Given KKT multipliers, split the active set:

$$\mathcal{A}^+ = \{\ell \in \mathcal{A} : \mu_\ell^\star > 0\} \text{ (strongly active)},$$
$$\mathcal{A}^0 = \{\ell \in \mathcal{A} : \mu_\ell^\star = 0\} \text{ (weakly active)}.$$

Critical cone.

$$C(\boldsymbol{x}^\star, \boldsymbol{\mu}^\star) = \left\{ \boldsymbol{d} : \begin{array}{ll} \nabla h_i(\boldsymbol{x}^\star)\boldsymbol{d} = 0, & i = 1, \dots, m, \\ \nabla g_\ell(\boldsymbol{x}^\star)\boldsymbol{d} = 0, & \ell \in \mathcal{A}^+, \\ \nabla g_\ell(\boldsymbol{x}^\star)\boldsymbol{d} \ge 0, & \ell \in \mathcal{A}^0 \end{array} \right\}$$

Equivalently $C = \{\boldsymbol{d} \in \mathcal{F}_L : \nabla f(\boldsymbol{x}^\star)\boldsymbol{d} = 0\}$ with $\mathcal{F}_L = \{\boldsymbol{d} : \nabla h_i \boldsymbol{d} = 0, \ \nabla g_\ell \boldsymbol{d} \ge 0, \ \ell \in \mathcal{A}\}$: no choice of multipliers is needed.

Conditions. SONC holds at a local minimizer with LICQ; SOSC at a KKT point gives a strict local minimizer. With $\boldsymbol{H}_L = \nabla^2_{xx}\mathcal{L}(\boldsymbol{x}^\star, \boldsymbol{\lambda}^\star, \boldsymbol{\mu}^\star)$:

$$\begin{cases} \boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} \ge 0, & \forall\, \boldsymbol{d} \in C \qquad \text{SONC} \\ \boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} > 0, & \forall\, \boldsymbol{d} \in C \setminus \{\boldsymbol{0}\} \qquad \text{SOSC} \end{cases}$$

Recipe.

1. Find the KKT point, then $\mathcal{A}$, $\mathcal{A}^+$, $\mathcal{A}^0$.
2. Strict complementarity ($\mathcal{A}^0 = \emptyset$): C is the kernel of the stacked gradients of equalities and active inequalities (the tangent space of the active constraints).
3. Take a basis $\boldsymbol{K}$ of that kernel and test $\boldsymbol{K}^T\boldsymbol{H}_L\boldsymbol{K}$ ($\succ 0$ gives SOSC, $\succeq 0$ is SONC); the verdict table of the Lagrange section applies.
4. Weak activity ($\mathcal{A}^0 \ne \emptyset$): parametrize the kernel of the equalities and of $\mathcal{A}^+$; keep the **sign constraints** $\nabla g_\ell \boldsymbol{d} \ge 0$, $\ell \in \mathcal{A}^0$; test $\boldsymbol{d}^T\boldsymbol{H}_L\boldsymbol{d}$ on that cone only.
5. If $C = \{\boldsymbol{0}\}$ (e.g. a vertex where $m + |\mathcal{A}| = n$ gradients are independent and all $\mu_\ell > 0$) SOSC holds trivially.
6. Interior point (no equalities, $\mathcal{A} = \emptyset$): $\nabla f(\boldsymbol{x}^\star) = \boldsymbol{0}$, $C = \mathbb{R}^n$, $\boldsymbol{H}_L = \nabla^2 f$; $\succ 0$ strict minimum, $\prec 0$ strict maximum, indefinite saddle, semidefinite singular inconclusive.

Watch out.

- Keep the sign constraints on $\mathcal{A}^0$: $\min -x^2 + y^2$ s.t. $x \geq 0$ at $(0,0)$ has $\mu^\star = 0$ and is positive definite on $\{d_1 = 0\}$, yet $\boldsymbol{d} = (1,0) \in C$ gives $\boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} = -2$. Conversely, under LICQ, $\boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} < 0$ for some $\boldsymbol{d}$ in the kernel of **all** active gradients does disprove minimality (that kernel is contained in C).
- Without LICQ the multipliers may be non-unique and SONC is not guaranteed.
- KKT stationarity is only necessary and SOSC is only sufficient. $\boldsymbol{H}_R \succeq 0$ proves nothing (flat directions, e.g. a whole segment of KKT points). A global statement needs a separate argument (compactness, bounds) or convexity: f convex, $\boldsymbol{h}$ affine, g_ℓ concave make every KKT point a global minimizer.

1.4 Summary

1. **Notation**

$$\nabla f,\ \nabla h_i,\ \nabla g_\ell \text{ row vectors}, \qquad \boldsymbol{J} = D\boldsymbol{h} \text{ (rows } \nabla h_i), \qquad \boldsymbol{H}_L = \nabla^2_{\boldsymbol{xx}} \mathcal{L}$$

2. **Problem: minimize**

$$f(\boldsymbol{x})$$

3. **Equality constraints:**

$$h_i(\boldsymbol{x}) = 0, \quad i = 1, \dots, m$$

4. **Inequality constraints:**

$$g_\ell(\boldsymbol{x}) \geq 0, \quad \ell = 1, \dots, p$$

5. **Lagrangian:**

$$\mathcal{L} = f - \sum_i \lambda_i h_i - \sum_\ell \mu_\ell g_\ell$$

1.4.1 Master problem

$$\begin{aligned} \text{minimize:} \quad & f(\boldsymbol{x}) \\ \text{subject to:} \quad & h_i(\boldsymbol{x}) = 0, \quad i = 1, \dots, m \\ & g_\ell(\boldsymbol{x}) \geq 0, \quad \ell = 1, \dots, p \end{aligned}$$

1.4.2 KKT system

$$\begin{cases} \nabla f - \sum_i \lambda_i \nabla h_i - \sum_\ell \mu_\ell \nabla g_\ell = \boldsymbol{0} & \text{stationarity} \\ h_i = 0 & \text{primal feasibility} \\ g_\ell \geq 0 & \text{primal feasibility} \\ \mu_\ell \geq 0 & \text{dual feasibility} \\ \mu_\ell\, g_\ell = 0 & \text{complementarity} \end{cases}$$

1.4.3 Tests

step	what to check
cases	for each guess of $\mathcal{A}$: $g_\ell = 0$ ($\ell \in \mathcal{A}$), $\mu_\ell = 0$ ($\ell \notin \mathcal{A}$); solve stationarity and $\boldsymbol{h} = \mathbf{0}$; discard any $g_\ell < 0$ or $\mu_\ell < 0$
CQ	LICQ: rank of $[\nabla h_i;\ \nabla g_\ell,\ \ell \in \mathcal{A}]$ is $m + \lvert\mathcal{A}\rvert$. MFCQ: ∇h_i independent and $\exists \boldsymbol{d}$: $\nabla h_i \boldsymbol{d} = 0$, $\nabla g_\ell \boldsymbol{d} > 0$ on $\mathcal{A}$. None: Fritz–John, direct argument
active sets	$\mathcal{A} = \{g_\ell = 0\}$, $\mathcal{A}^+ = \{\mu_\ell > 0\}$, $\mathcal{A}^0 = \{\mu_\ell = 0\}$
critical cone	$\nabla h_i \boldsymbol{d} = 0$; $\nabla g_\ell \boldsymbol{d} = 0$ on $\mathcal{A}^+$; $\nabla g_\ell \boldsymbol{d} \geq 0$ on $\mathcal{A}^0$
reduced Hessian	equalities only, or $\mathcal{A}^0 = \emptyset$: $\boldsymbol{H}_R = \boldsymbol{K}^T \boldsymbol{H}_L \boldsymbol{K}$, $\boldsymbol{J}_{\text{act}} \boldsymbol{K} = \mathbf{0}$, with $\boldsymbol{J}_{\text{act}}$ the stacked ∇h_i and ∇g_ℓ, $\ell \in \mathcal{A}$
interior	no equalities, $\mathcal{A} = \emptyset$: $\nabla f = \mathbf{0}$, $\mathcal{C} = \mathbb{R}^n$, $\boldsymbol{H}_L = \nabla^2 f$
SONC	$\boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} \geq 0$ on $\mathcal{C}$ ($\boldsymbol{H}_R \succeq 0$ if $\mathcal{C}$ is the kernel); needs LICQ
SOSC	$\boldsymbol{d}^T \boldsymbol{H}_L \boldsymbol{d} > 0$ on $\mathcal{C} \setminus \{\mathbf{0}\}$ ($\boldsymbol{H}_R \succ 0$ if $\mathcal{C}$ is the kernel); strict local minimizer
definiteness	$\succ 0$: leading minors $\Delta_k > 0$; $\succeq 0$: **all** principal minors ≥ 0; or eigenvalues

2. DAE

Everything needed to find the index of a DAE, reduce it to an ODE plus invariants, and set up constrained mechanics. Numerical methods (consistent initialization, weighted projection, impact re-initialization, order reduction) are in the companion volume *Numerical Methods*.

Colors: differential equations, algebraic equations, constraints, hidden constraints, initial and consistency conditions, labels.

2.1 Linear DAEs

Problem.

$$E\dot{x}(t) = Ax(t) + f(t), \qquad E, A \in \mathbb{R}^{n\times n} \text{ constant} \tag{2.1}$$

E	meaning	initial data
$\det E \neq 0$	ODE $\dot{x} = E^{-1}Ax + E^{-1}f$, index 0 (piecewise continuous f: unique solution)	all of $x(0)$ free
$E = 0$	algebraic system $Ax = -f$	A invertible: $x(0)$ fixed by $f(0)$
singular, $\neq 0$	genuine DAE, possible hidden constraints	depends on the pencil

Regular pencil. (E, A) is regular iff $\det(A - \lambda_0 E) \neq 0$ for some $\lambda_0 \in \mathbb{R}$. A singular pencil loses uniqueness or existence.

Weierstrass form. The pencil is regular iff there are invertible P, Q with

$$(PEQ,\ PAQ) = \left(\begin{bmatrix} N & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & J \end{bmatrix} \right)$$

- $\boldsymbol{N}$ upper triangular nilpotent (any upper triangular matrix with zero diagonal is nilpotent), $\boldsymbol{N}^{\nu} = \boldsymbol{0}$, ν the smallest such integer (the *degree*): blocks at the eigenvalue 0 of $\boldsymbol{E}$ (eigenvalues at infinity, algebraic part).
- $\boldsymbol{J}$ in Jordan form: finite generalized eigenvalues (differential part).

Decoupling. With $\boldsymbol{z} = \boldsymbol{Q}^{-1}\boldsymbol{x} = (\boldsymbol{z}_1; \boldsymbol{z}_2)$ and $\boldsymbol{g} = \boldsymbol{P}\boldsymbol{f} = (\boldsymbol{g}_1; \boldsymbol{g}_2)$:

$$\begin{cases} \boldsymbol{N}\dot{\boldsymbol{z}}_1 = \boldsymbol{z}_1 + \boldsymbol{g}_1(t) & \text{algebraic block} \\ \dot{\boldsymbol{z}}_2 = \boldsymbol{J}\boldsymbol{z}_2 + \boldsymbol{g}_2(t) & \text{differential block} \end{cases}$$

$$\begin{cases} \boldsymbol{z}_2(t) = \mathrm{e}^{\boldsymbol{J}t}\boldsymbol{z}_2(0) + \displaystyle\int_0^t \mathrm{e}^{\boldsymbol{J}(t-\tau)}\boldsymbol{g}_2(\tau)\,\mathrm{d}\tau & \text{variation of constants} \\ \boldsymbol{z}_1(t) = -\displaystyle\sum_{k=0}^{\nu-1} \boldsymbol{N}^k \boldsymbol{g}_1^{(k)}(t) & \text{algebraic solution} \\ \boldsymbol{x}(t) = \boldsymbol{Q}\boldsymbol{z}(t) \end{cases}$$

Index. Kronecker index $= \nu$ (the largest nilpotent Jordan block); $\boldsymbol{E}$ invertible means index 0. The highest derivative of $\boldsymbol{g}_1$ needed is $\nu - 1$.

index	nilpotency	highest derivative of the forcing
0	no algebraic block	not applicable
1	$\boldsymbol{N} = \boldsymbol{0}$	0
2	$\boldsymbol{N}^2 = \boldsymbol{0}$	1
ν	$\boldsymbol{N}^{\nu} = \boldsymbol{0}$	$\nu - 1$

Solvability and consistent initial values. The problem with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ has a solution if (a) $(\boldsymbol{E}, \boldsymbol{A})$ is regular; (b) $\boldsymbol{g}_2$ is locally integrable (piecewise continuous suffices) and $\boldsymbol{g}_1$ has the derivatives through order $\nu - 1$ (more smoothness for a classical C^1 solution); (c) $\boldsymbol{x}_0$ is consistent:

$$\boldsymbol{x}_0 = \boldsymbol{Q}\begin{bmatrix} -\displaystyle\sum_{k=0}^{\nu-1} \boldsymbol{N}^k \boldsymbol{g}_1^{(k)}(0) \\ \boldsymbol{z}_2(0) \end{bmatrix} \qquad \text{algebraic block fixed, differential block free}$$

Recipe (typical exercise: analyze $\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}$, find the index, reduce).

1. $\det \boldsymbol{E} \neq 0$: ODE, index 0, stop.
2. Test regularity, $\det(\boldsymbol{A} - \lambda\boldsymbol{E}) \not\equiv 0$; otherwise uniqueness or existence is lost.
3. Bring the pencil to Weierstrass form and read $\boldsymbol{N}$, $\boldsymbol{J}$.
4. Index $= \nu$, the smallest integer with $\boldsymbol{N}^{\nu} = \boldsymbol{0}$.
5. Solve the differential block by variation of constants and the algebraic block by $\boldsymbol{z}_1 = -\sum_{k=0}^{\nu-1} \boldsymbol{N}^k \boldsymbol{g}_1^{(k)}$; then $\boldsymbol{x} = \boldsymbol{Q}\boldsymbol{z}$.
6. Build a consistent $\boldsymbol{x}_0$ (algebraic block fixed, $\boldsymbol{z}_2(0)$ free) and check that $\boldsymbol{g}_1$ has $\nu - 1$ derivatives.

Watch out.

- Sign convention: here $\boldsymbol{E}\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{f}$ (Equation 2.1). The next section (Gauss reduction) uses $\boldsymbol{E}\boldsymbol{x}' + \boldsymbol{A}\boldsymbol{x} = \boldsymbol{f}$ (Equation 2.2), the opposite sign of $\boldsymbol{A}$; the regularity test there reads $\det(\boldsymbol{A} + \lambda_0\boldsymbol{E}) \neq 0$.

- Algebraic blocks written as $N\dot{z} + z = g$ (opposite to the decoupled equation above) have solution $z = \sum_{k=0}^{\nu-1} (-1)^k N^k g^{(k)}$.
- Singular E does **not** imply a singular pencil. With a singular pencil the solution is not unique: $E = \operatorname{diag}(1,0)$, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = 0$ reads $\dot{x}_1 = x_2$; with $x(0) = 0$ every C^1 function φ, $\varphi(0) = 0$, gives a solution $x_2 = \varphi$, $x_1 = \int_0^t \varphi$.
- For $\nu \geq 1$ an arbitrary $x(0)$ is not consistent: check (c) before claiming a solution.

2.2 Index Reduction for Linear DAEs

Jordan forms are not numerically stable; Gaussian elimination gives the same structure.

Problem (sign of A opposite to the previous section).

$$Ex' + Ax = f(t), \qquad E, A \in \mathbb{R}^{n\times n} \tag{2.2}$$

Rank-revealing LU of E, $k = \operatorname{rank} E$ (L unit lower triangular $k \times k$, U upper triangular $k \times k$ with nonzero diagonal):

$$P_L E P_R = \begin{pmatrix} L & 0 \\ M & I \end{pmatrix} \begin{pmatrix} U & K \\ 0 & 0 \end{pmatrix},$$

$$Q^T = \begin{pmatrix} L^{-1} & 0 \\ -ML^{-1} & I \end{pmatrix} P_L \;\Rightarrow\; Q^T E = \begin{pmatrix} U & K \\ 0 & 0 \end{pmatrix} P_R^T$$

The last $n - k$ rows of $Q^T E$ vanish. Write $Q^T = (Q_d^T; Q_a^T)$, so $Q_a^T E = 0$ (Q_a is a basis of $\mathcal{N}(E^T)$). No explicit null-space basis is needed.

Recipe (loop *separate, file, differentiate*; each pass lowers the index by one).

1. If $\det E \neq 0$: ODE, index 0, stop.
2. Factorize E and apply the **same** Q^T to A and f:

$$\begin{cases} E_d x' + A_d x = f_d & \text{differential rows} \\ A_a x = f_a(t) & \text{algebraic rows} \end{cases}$$

   with $E_d = Q_d^T E$, $A_d = Q_d^T A$, $f_d = Q_d^T f$, $A_a = Q_a^T A$, $f_a = Q_a^T f$.
3. File the algebraic rows as hidden constraints: $A_a x = f_a$ must hold for $x(0)$.
4. Differentiate them, $A_a x' = f_a'$, and stack under the differential rows:

$$\widetilde{E} = \begin{pmatrix} E_d \\ A_a \end{pmatrix}, \quad \widetilde{A} = \begin{pmatrix} A_d \\ 0 \end{pmatrix}, \quad \widetilde{f} = \begin{pmatrix} f_d \\ f_a' \end{pmatrix}$$

   Return to step 1 with $\widetilde{E}, \widetilde{A}, \widetilde{f}$.
5. When $\widetilde{E}$ is nonsingular: ODE $x' = -\widetilde{E}^{-1}\widetilde{A}x + \widetilde{E}^{-1}\widetilde{f}$. Index = number of differentiations (step 4); for a regular pencil it equals the nilpotency degree ν.
6. The initial condition must satisfy the algebraic equations of **every** pass (the original ones and the hidden ones): $x(0)$ is only partly free.

Invariants. $\Psi(x,t)$ is an invariant (hidden constraint) of $x' = F(x,t)$ iff along solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(x(t),t) = \frac{\partial \Psi}{\partial x} F(x,t) + \frac{\partial \Psi}{\partial t} \equiv 0$$

so its value is fixed by the initial condition. Index reduction never discards algebraic equations: it files them as invariants of the resulting ODE, which has **more** solutions than the DAE.

Watch out.

- The rank of $\widetilde{E}$ need not increase after one differentiation: keep the differential rows E_d (as in step 4). Rank progression alone is not an index theorem; the loop gives the Kronecker index only under regularity and rank assumptions.
- The separation is by *type* of equation, not by variable: A_a still couples all unknowns.
- ODEs have invariants too ($x^2 + y^2$ for $\dot{x} = y$, $\dot{y} = -x$) and numerical methods do not preserve them: explicit Euler gives $\|v_{k+1}\| \geq \|v_k\|$, implicit Euler $\|v_{k+1}\| \leq \|v_k\|$, for every step size. The remedy is projection, Baumgarte stabilization or a structure-preserving method, not a smaller step.

2.3 The Differential Index

Problem. $F(x, x', t) = 0$, $x \in \mathbb{R}^n$, with $F_{x'}$ singular (some combinations of equations contain no x').

Derivative array. A system of $(q+1)n$ equations in $x', \ldots, x^{(q+1)}$ for fixed (x, t):

$$\mathcal{F}_q(x, x', \ldots, x^{(q+1)}, t) = \begin{pmatrix} F \\ \frac{\mathrm{d}}{\mathrm{d}t}F \\ \vdots \\ \frac{\mathrm{d}^q}{\mathrm{d}t^q}F \end{pmatrix} = 0$$

Differentiation index. The least $q \geq 0$ such that $\mathcal{F}_q = 0$ determines x' locally and uniquely as a function of (x, t), i.e. yields the underlying ODE $x' = G(x, t)$. An explicit ODE has index 0. It presumes constant rank of the Jacobians involved; otherwise the index may be undefined or vary along a trajectory.

Recipe on a given system.

1. Write the derivatives of **all** equations row by row, up to order q (row k is the k-th time derivative of each equation).
2. Increase q until x' is determined for **every** component of x; eliminate $x'', \ldots, x^{(q+1)}$ by substitution and linear combinations.
3. Classify the rows: used for the ODE (they give $x' = G$); invariants (relations between x and t only: the original algebraic equations and the hidden ones; they characterize the consistent initial values); *discarded* (not needed, they hold automatically).
4. The DAE is equivalent to the underlying ODE **together with** its invariants, not to the ODE alone.

Table 2.1: Index by the derivative array.

system	index q	invariants	underlying ODE
$x' = -x$	0	none	$x' = -x$
$x' = -y,\ 0 = x + y - \sin t$	1	$x + y - \sin t = 0$	$x' = -y,\ y' = y + \cos t$
$0 = x - \sin t,\ 0 = x' + y$	2	$x - \sin t = 0,\ y + \cos t = 0$	$x' = \cos t,\ y' = \sin t$

system	index q	invariants	underlying ODE
$x' = y,\ 0 = x$	2	$x = 0,\ y = 0$	$x' = y,\ y' = 0$
$x' = y,\ y' = z,\ 0 = x$	3	$x = 0,\ y = 0,\ z = 0$	$x' = y,\ y' = z,\ z' = 0$

Other notions of index (they agree on important regular classes, not in general).

- *Perturbation index:* how derivatives of a residual perturbation enter the error estimate; depends on the perturbation model and norms.
- *Tractability index:* a sequence of projectors separating differential and algebraic directions.
- *Strangeness index:* derivative arrays and rank-revealing transformations; also for time-varying and over- or underdetermined systems.
- *Differentiation vs. Kronecker/nilpotency index:* equal for a regular constant-coefficient pencil (ODE has index 0); do not transfer to singular pencils or nonlinear DAEs without a separate theorem.

For a regular constant-coefficient pencil of Kronecker index $\nu \geq 1$:

Table 2.2: Index notions for a regular pencil, $\nu \geq 1$ (standard results of the respective theories, not derived here; conventions for $\nu = 0$ differ).

differentiation	perturbation	tractability	strangeness
ν	ν	ν	$\nu - 1$

Watch out. The index is a property of the specified formulation; rewriting $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{x}', t) = \boldsymbol{0}$ may change it (see Section 2.4).

2.4 Semi-Explicit Form and Its Classification

Problem ($\boldsymbol{x}$ differential variables, $\boldsymbol{y}$ algebraic variables; write $\boldsymbol{g}_y = \partial \boldsymbol{g}/\partial \boldsymbol{y}$, etc.).

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t), \qquad \boldsymbol{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, t)$$

From implicit form. For $\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{x}', t) = \boldsymbol{0}$: if $\boldsymbol{F}_{\boldsymbol{x}'}$ is nonsingular, the implicit function theorem gives an ODE $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, t)$. Otherwise set $\boldsymbol{v} = \boldsymbol{x}'$:

$$\boldsymbol{x}' = \boldsymbol{v}, \qquad \boldsymbol{0} = \boldsymbol{F}(\boldsymbol{x}, \boldsymbol{v}, t)$$

Implicit function theorem. If $\boldsymbol{g}_y$ is nonsingular at $\boldsymbol{g} = \boldsymbol{0}$, locally $\boldsymbol{y} = \boldsymbol{\phi}(\boldsymbol{x}, t)$ with $\boldsymbol{\phi}_x = -\boldsymbol{g}_y^{-1}\boldsymbol{g}_x$, $\boldsymbol{\phi}_t = -\boldsymbol{g}_y^{-1}\boldsymbol{g}_t$.

Index tests. Which Jacobian (or product) must be nonsingular:

Table 2.3: Index classification of semi-explicit DAEs (z is the algebraic variable of the last row).

index	system	nonsingular matrix	invariants of the reduced ODE
1	$x' = f(x,y,t),\ 0 = g(x,y,t)$	g_y	$g = 0$
2	$x' = f(x,y,t),\ 0 = g(x,t)$	$g_x f_y$	$g = 0,\ h := g_x f + g_t = 0$
3	$x' = f(x,y,z,t),\ y' = g(x,y,t),$ $0 = h(y,t)$	$h_y g_x f_z$	$h = 0,\ a := h_y g + h_t = 0,\ \dot{a} = 0$

Recipe (read the index without reducing).

1. Differentiate the constraint along the flow, substituting $x' = f$ (and $y' = g$ if present).
2. If the algebraic variable is absent from the result, differentiate again.
3. Stop at the first of $g, \dot{g}, \ddot{g}, \ldots$ that contains the algebraic variable with a nonsingular Jacobian. Index = (number of differentiations of the constraint) +1.
4. Index 1: one more differentiation gives $y' = -g_y^{-1}(g_t + g_x f)$. Index 2: h has $h_y = g_x f_y$.

Hessenberg form of size r.

$$\begin{cases} \dot{x}_i = f_i(x_1, \ldots, x_{i+1}, t), & i = 1, \ldots, r-1 \\ 0 = g(x_1, t) \end{cases}$$

$$\frac{\partial g}{\partial x_1}\frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_3}\cdots\frac{\partial f_{r-1}}{\partial x_r} \text{ nonsingular} \implies \text{differentiation index} = r$$

$r = 2$ is the index-2 test; $r = 3$ is the index-3 test with $x_1 = y$, $x_2 = x$, $x_3 = z$. Index 1 is the degenerate case $0 = g(x_1, x_2, t)$ with g_{x_2} nonsingular.

Constrained mechanics is Hessenberg of size 3 ($x_1 = q$, $x_2 = v$, $x_3 = \lambda$; $G = \partial\Phi/\partial q$):

$$\begin{cases} \dot{q} = v \\ M(q)\dot{v} = F(q,v,t) - G(q)^T\lambda \\ 0 = \Phi(q) \end{cases} \qquad G\,I\,(-M^{-1}G^T) = -GM^{-1}G^T$$

For M positive definite the chain is nonsingular iff G has full row rank: then the system has index 3. Here λ has the opposite sign to Equation 2.4, and $G = \Phi_q$ is unrelated to $G(x,t)$ of the general DAE.

Watch out.

- $v = x'$ is not an index-preserving simplification: it typically raises the index by one.
- Index 2 has two invariants, each weaker than the previous one: a numerical trajectory can satisfy $h = 0$ to machine precision and still drift from $g = 0$.
- For consistent initial values at index 1, solve $g(x_0, y_0, t_0) = 0$ for y_0; this fails where g_y is singular. At index 2 choose x_0 with $g(x_0, t_0) = 0$ and solve $h(x_0, y_0, t_0) = 0$ for y_0 (implicit function theorem, $h_y = g_x f_y$).

2.5 General Index Reduction

Problem. Covers every semi-explicit system ($E = \mathrm{diag}(I, 0)$) and the linear DAE $Ex' + Ax = f$ (Equation 2.2; E constant, $G = f - Ax$).

$$E(x,t)\,\dot{x} = G(x,t), \qquad E : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times n} \tag{2.3}$$

Table 2.4: Cases for Equation 2.3.

case	consequence
$E(x,t)$ nonsingular	ODE $\dot{x} = E^{-1}G$, index 0 (factorize, do not invert)
$E \equiv 0$	$0 = G(x,t)$; if G_x is nonsingular, $\dot{x} = -G_x^{-1}G_t$; $G = 0$ is an invariant; $x(0)$ must solve $G(x(0),0) = 0$
E singular, $\neq 0$	genuine DAE: the rank may depend on (x,t), every statement is local

Row-only rank-revealing factorization (P permutation, L unit lower triangular, M_1 of full row rank $k = \operatorname{rank} E$):

$$PE = L\begin{pmatrix} M_1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad T = L^{-1}P, \qquad TE = \begin{pmatrix} M_1 \\ 0 \end{pmatrix}$$

Multiplying the DAE by T keeps the solution set and separates the algebraic rows. Recompute the factorization at every pass; never form L^{-1} (forward substitution). By hand: row operations on the table $[E \mid G]$ (swap, add a multiple of a row, scale) until the zero rows of E appear.

Recipe (nonlinear index reduction).

1. Matrix form $E(x,t)x' = G(x,t)$. An implicit DAE $f(x,x',t) = 0$ is first written in semi-explicit form with $y = x'$ (Section 2.4).
2. If $\det E \neq 0$: ODE, index 0, stop.
3. Separate with $Q^T = (Q_d^T; Q_a^T)$: Q_a a basis of $\mathcal{N}(E^T)$ ($Q_a^T E = 0$), Q_d completing it so that $Q = (Q_d, Q_a)$ is nonsingular (equivalently the last rows of T give Q_a^T):

$$\begin{cases} E_d x' = G_d & \text{differential rows} \\ \quad 0 = G_a(x,t) & \text{algebraic rows (invariant)} \end{cases}$$

   with $E_d = Q_d^T E$, $G_d = Q_d^T G$, $G_a = Q_a^T G$.
4. Differentiate the algebraic rows, $G_a^t + G_a^x x' = 0$ (superscripts: partial derivatives in t and x):

$$\widetilde{E}x' = \widetilde{G}, \qquad \widetilde{E} = \begin{pmatrix} E_d \\ -G_a^x \end{pmatrix}, \quad \widetilde{G} = \begin{pmatrix} G_d \\ G_a^t \end{pmatrix}$$

5. Repeat from step 2 until $\widetilde{E}$ is invertible. Index = number of times step 4 was performed. The equations $G_a = 0$ of every pass are kept as invariants.

Watch out.

- Requires $E(x,t)$ of **constant rank** on the domain; otherwise the index changes along the trajectory.
- The permutation is not cosmetic: P depends on (x,t) through the pivot choice, and the "zero" rows are zero only within a tolerance. The separation is a numerical rank decision; near a point where the rank of E changes no factorization makes the choice for you.

2.6 Dummy Variables

Differentiated constraints replace the original ones (stored as invariants) instead of enlarging the system.

Requirements. Semi-explicit form with differential variables $x_d \in \mathbb{R}^{n_d}$ and algebraic variables $x_a \in \mathbb{R}^{n_a}$, $n_d + n_a = n$:

$$x_d' = f(x_d, x_a, t), \qquad 0 = g(x_d, x_a, t)$$

Regularity condition: x_a appears **linearly**, $f = f_0(x_d, t) + A(x_d, t)x_a$, $g = g_0(x_d, t) + B(x_d, t)x_a$. Algebraic variables are *linear* x_l (linear or absent from the constraints) or *nonlinear* x_n.

Preprocessing.

1. Implicit DAE: variables whose derivative appears are differential, the others algebraic; each derivative occurring inside an equation is replaced by a new algebraic variable w with $x' = w$.
2. If $x_n \neq \emptyset$: introduce $x_n' = v_n$; reclassify as new differential (x_d, x_n), new algebraic (x_l, v_n).
3. The DAE now has the form

$$x_d' = f(x_d, t) + A(x_d, t)x_a, \qquad 0 = g(x_d, t) + B(x_d, t)x_a$$

Recipe (reduction on $B(x_d, t)x_a = -g(x_d, t)$).

1. If $\det B \neq 0$: **index 1**, $x_a = -B^{-1}g$, stop. Otherwise continue.
2. Let $r = \operatorname{rank} B$, Q_a a basis of $\mathcal{N}(B^T)$ ($Q_a^T B = 0$), Q_d completing $Q = (Q_d, Q_a)$ to a nonsingular matrix. Multiply the constraints by Q^T:

$$\begin{cases} B_d x_a = -g_d & \text{part with algebraic variables} \\ g_a(x_d, t) = 0 & \text{part free of algebraic variables, hidden constraint} \end{cases}$$

   with $B_d = Q_d^T B$, $g_d = Q_d^T g$, $g_a = Q_a^T g$.
3. Differentiate g_a and substitute $x_d' = f + Ax_a$:

$$\frac{\partial g_a}{\partial x_d} A\, x_a = -\left(\frac{\partial g_a}{\partial t} + \frac{\partial g_a}{\partial x_d} f\right) \qquad \text{new equation, linear in the algebraic variables}$$

   Stack it below $B_d x_a = -g_d$:

$$\widetilde{B} x_a = -\widetilde{g}, \qquad \widetilde{B} = \begin{pmatrix} B_d \\ \dfrac{\partial g_a}{\partial x_d} A \end{pmatrix}, \quad \widetilde{g} = \begin{pmatrix} g_d \\ \dfrac{\partial g_a}{\partial t} + \dfrac{\partial g_a}{\partial x_d} f \end{pmatrix}$$

   Go to step 1 with $\widetilde{B}, \widetilde{g}$.
4. Differentiation index = (number of differentiations, step 3) +1.
5. At index 1: $x_a = -\widetilde{B}^{-1}\widetilde{g}$ (solve, do not invert) and $x_d' = f + Ax_a$. The removed $g_a = 0$ of every pass are the hidden constraints: $x_d(0)$ must satisfy them.
6. Derivatives of the algebraic unknowns (for implicit solvers), for p any of t or a component of x_d: differentiate $\widetilde{B}x_a = -\widetilde{g}$,

$$\widetilde{B}\,\partial_p x_a = -\partial_p \widetilde{g} - (\partial_p \widetilde{B})x_a$$

Choosing what to differentiate (several constraints). Build the 0/1 dependency table (which algebraic variables each constraint contains). Differentiate only constraints that contain **no** algebraic variable (the rows of $\boldsymbol{g}_a$): they lower the index without introducing derivatives of algebraic variables. If none exists and $\boldsymbol{B}$ is singular, combine constraints (step 2) to eliminate algebraic variables.

Watch out.

- Structural rank of the 0/1 table (e.g. `sprank`) below its size proves $\boldsymbol{B}$ singular; full structural rank does not prove it nonsingular.
- Without the linearity of $\boldsymbol{x}_a$ the algorithm does not apply: preprocess first.
- Keep the original constraints as invariants: the reduced system alone does not enforce them.
- Drift and Baumgarte stabilization of these constraints: see the Remedy in Section 2.7.

2.7 DAEs from Lagrangian Mechanics

Lagrangian. Generalized coordinates $\boldsymbol{q} \in \mathbb{R}^n$, kinetic energy T, potential energy U, mass matrix $\boldsymbol{M}(\boldsymbol{q})$ symmetric positive definite:

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - U(\boldsymbol{q}, t), \qquad T = \tfrac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} = \tfrac{1}{2}\sum_{j,k} m_{jk}(\boldsymbol{q})\dot{q}_j\dot{q}_k$$

Euler–Lagrange equations (stationary action, fixed endpoints; $\boldsymbol{\tau}$ = non-conservative generalized forces, $\boldsymbol{\tau} = \boldsymbol{0}$ if absent):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{\tau}$$

Equations of motion (quadratic kinetic energy), componentwise and in matrix form:

$$\sum_{j=1}^{n} m_{ij}(\boldsymbol{q})\ddot{q}_j + \sum_{j,k=1}^{n} h_{ijk}(\boldsymbol{q})\dot{q}_j\dot{q}_k + g_i(\boldsymbol{q}) = \tau_i$$

$$h_{ijk} = \frac{\partial m_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial m_{jk}}{\partial q_i}, \qquad g_i = \frac{\partial U}{\partial q_i}$$

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}, \qquad (\boldsymbol{C}\dot{\boldsymbol{q}})_i = \sum_{j,k} h_{ijk}\dot{q}_j\dot{q}_k$$

$\boldsymbol{C}$ collects centrifugal and Coriolis effects.

Constrained mechanics. Holonomic constraints $\boldsymbol{\phi}(\boldsymbol{q}, t) = \boldsymbol{0}$, $\boldsymbol{\phi} \in \mathbb{R}^m$, multipliers $\boldsymbol{\lambda} \in \mathbb{R}^m$; write $\boldsymbol{\phi}_{\boldsymbol{q}} = \partial\boldsymbol{\phi}/\partial\boldsymbol{q}$. The modified action $\bar{S} = \int_{t_1}^{t_2} \left(L + \boldsymbol{\lambda}^T\boldsymbol{\phi}\right)\mathrm{d}t$ is stationary (endpoints satisfy the constraints):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\dfrac{\partial L}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{\phi}_{\boldsymbol{q}}^T\boldsymbol{\lambda} \;\; (+\boldsymbol{\tau}) & \text{dynamics} \\ \boldsymbol{\phi}(\boldsymbol{q}, t) = \boldsymbol{0} & \text{position constraint} \end{cases} \tag{2.4}$$

If rank $\boldsymbol{\phi}_{\boldsymbol{q}} = m$ there are exactly $n - m$ degrees of freedom. The term $\boldsymbol{\phi}_{\boldsymbol{q}}^T\boldsymbol{\lambda}$ is the constraint force.

First-order form of Equation 2.4, $\boldsymbol{v} = \boldsymbol{q}'$, $\boldsymbol{n} = \boldsymbol{\tau} - C(\boldsymbol{q}, \boldsymbol{v})\boldsymbol{v} - \boldsymbol{g}(\boldsymbol{q})$:

$$\begin{cases} \boldsymbol{q}' = \boldsymbol{v} & \text{kinematics} \\ M(\boldsymbol{q}, \boldsymbol{v}, t)\boldsymbol{v}' - \boldsymbol{\phi}_{\boldsymbol{q}}^T \boldsymbol{\lambda} = \boldsymbol{n}(\boldsymbol{q}, \boldsymbol{v}, t) & \text{dynamics} \\ \boldsymbol{\phi}(\boldsymbol{q}, t) = \boldsymbol{0} & \text{position constraint} \end{cases}$$

Recipe (ODE reduction).

1. Differentiate the constraint once: velocity-level hidden constraint
$$\boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{v} + \boldsymbol{\phi}_t = \boldsymbol{0}$$
(time-independent constraint: $\boldsymbol{v}$ is orthogonal to the constraint gradients).
2. Differentiate again: acceleration-level hidden constraint
$$\boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{v}' + \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}, t) = \boldsymbol{0}, \qquad \boldsymbol{m} = \frac{\partial}{\partial \boldsymbol{q}}\left(\boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{v} + \boldsymbol{\phi}_t\right)\boldsymbol{v} + \frac{\partial}{\partial t}\left(\boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{v} + \boldsymbol{\phi}_t\right)$$
$\boldsymbol{m}$ collects the terms without $\boldsymbol{v}'$; for time-independent $\boldsymbol{\phi}$, $\boldsymbol{m} = \frac{\partial^2 \boldsymbol{\phi}}{\partial \boldsymbol{q}^2}[\boldsymbol{v}, \boldsymbol{v}]$.
3. Replace $\boldsymbol{\phi} = \boldsymbol{0}$ by step 2:
$$\begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{M} & -\boldsymbol{\phi}_{\boldsymbol{q}}^T \\ \boldsymbol{0} & -\boldsymbol{\phi}_{\boldsymbol{q}} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}' \\ \boldsymbol{v}' \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{n}(\boldsymbol{q}, \boldsymbol{v}, t) \\ \boldsymbol{m}(\boldsymbol{q}, \boldsymbol{v}, t) \end{bmatrix}$$
4. The matrix is invertible iff $\boldsymbol{D} = \boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{M}^{-1}\boldsymbol{\phi}_{\boldsymbol{q}}^T$ is nonsingular ($\boldsymbol{M}$ positive definite); guaranteed if rank $\boldsymbol{\phi}_{\boldsymbol{q}} = m$ for all $\boldsymbol{q}, t$. Then
$$\boldsymbol{\lambda} = -\boldsymbol{D}^{-1}\left(\boldsymbol{m} + \boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{M}^{-1}\boldsymbol{n}(\boldsymbol{q}, \boldsymbol{v}, t)\right)$$
and the system is an explicit ODE in $(\boldsymbol{q}, \boldsymbol{v})$.
5. Consistent initial values: $\boldsymbol{\phi}(\boldsymbol{q}_0, t_0) = \boldsymbol{0}$ and $\boldsymbol{\phi}_{\boldsymbol{q}}\boldsymbol{v}_0 + \boldsymbol{\phi}_t = \boldsymbol{0}$; $\boldsymbol{\lambda}(t_0)$ then follows from step 4.
6. Index: two differentiations determine $\boldsymbol{\lambda}$ algebraically (index-1 DAE), a third gives $\boldsymbol{\lambda}'$: differential index 3.

Pattern: planar pendulum ($\boldsymbol{q} = (x, y)^T$, y up, $\boldsymbol{\phi} = x^2 + y^2 - \ell^2$, $\boldsymbol{\phi}_{\boldsymbol{q}} = 2\boldsymbol{q}^T$, $\boldsymbol{m} = 2\boldsymbol{v}^T\boldsymbol{v}$):

$$\begin{cases} m\ddot{x} = 2x\lambda, \quad m\ddot{y} = 2y\lambda - mg & \text{dynamics} \\ x^2 + y^2 = \ell^2 & \text{constraint} \\ 2\boldsymbol{q}^T\boldsymbol{v} = 0 & \text{velocity constraint} \\ 2\boldsymbol{v}^T\boldsymbol{v} + 2\boldsymbol{q}^T\boldsymbol{v}' = 0 & \text{acceleration constraint} \end{cases} \qquad \lambda = -\frac{m}{2\ell^2}\left(\boldsymbol{v}^T\boldsymbol{v} - g q_2\right)$$

In the minimal coordinate θ: $\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$.

Constraint drift. The ODE reduction replaces $\boldsymbol{\phi} = \boldsymbol{0}$ by $\ddot{\boldsymbol{\phi}} = \boldsymbol{0}$: the fictitious constraint

$$\widetilde{\boldsymbol{\phi}}(\boldsymbol{q}, t) = \boldsymbol{\phi}(\boldsymbol{q}, t) + \boldsymbol{a}_0 + \boldsymbol{a}_1 t, \qquad \frac{\mathrm{d}^2\boldsymbol{\phi}}{\mathrm{d}t^2} = \frac{\mathrm{d}^2\widetilde{\boldsymbol{\phi}}}{\mathrm{d}t^2}$$

has the same reduced system. Hence the original constraint is violated with an error growing in time, at best linearly (the term $\boldsymbol{a}_1 t$ has no effect on the reduced system), possibly faster because of numerical errors.

Remedy (Baumgarte stabilization). Replace the acceleration constraint $\ddot{\boldsymbol{\phi}} = \mathbf{0}$ by, with tuning parameters $\eta, \omega > 0$,

$$\ddot{\boldsymbol{\phi}} + 2\eta\omega\,\dot{\boldsymbol{\phi}} + \omega^2\boldsymbol{\phi} = \mathbf{0} \quad \text{i.e.} \quad \boldsymbol{\phi}_q\boldsymbol{v}' + \boldsymbol{m} + 2\eta\omega(\boldsymbol{\phi}_q\boldsymbol{v} + \boldsymbol{\phi}_t) + \omega^2\boldsymbol{\phi} = \mathbf{0}$$

In the matrix equation of step 3 and in the formula for $\boldsymbol{\lambda}$, $\boldsymbol{m}$ becomes $\boldsymbol{m} + 2\eta\omega(\boldsymbol{\phi}_q\boldsymbol{v} + \boldsymbol{\phi}_t) + \omega^2\boldsymbol{\phi}$. Other remedies: projection onto the position and velocity constraints, structure-preserving integrators (companion volume).

Baumgarte for any constraint. If a constraint $c = 0$ was replaced by its k-th derivative $c^{(k)} = 0$ in a reduction, replace that row by a stable combination of $c, \dots, c^{(k)}$ (the degree equals the number of differentiations; $\tau, \eta, \omega > 0$):

$$c' + \frac{c}{\tau} = 0 \;\; (k = 1), \qquad c'' + 2\eta\omega\,c' + \omega^2 c = 0 \;\; (k = 2)$$

For larger k use a polynomial of degree k in $\mathrm{d}/\mathrm{d}t$ with all roots in the left half-plane, e.g. $(\mathrm{d}/\mathrm{d}t + \alpha)^k c = 0$, $\alpha > 0$. In the dummy-variable system $\widetilde{\boldsymbol{B}}\boldsymbol{x}_a = -\widetilde{\boldsymbol{g}}$ with $k = 1$, the entries of $\widetilde{\boldsymbol{g}}$ from the differentiated row become $\frac{\partial \boldsymbol{g}_a}{\partial t} + \frac{\partial \boldsymbol{g}_a}{\partial \boldsymbol{x}_d}\boldsymbol{f} + \boldsymbol{g}_a/\tau$.

Watch out.

- Drift is intrinsic to the substitution, not a defect of a particular integrator or of insufficient accuracy: the reduced system has no mechanism that pulls the solution back onto the constraint.
- Baumgarte stabilization only damps the violation: η, ω are tuned, and too large values give stiffness and artificial damping.
- Baumgarte needs a genuine chain $c, c', c'', \dots$ (one constraint and its total derivatives along solutions, rows differentiated as they stand). After row recombination (Gauss route, or step 2 of the dummy-variable recipe) the algebraic rows of later passes are in general not the derivatives of the earlier ones and cannot be inserted in the stabilizing equation.
- $\boldsymbol{\lambda}$ is not free: $\boldsymbol{D}$ must be nonsingular, i.e. linearly independent constraint gradients.

2.8 Summary

1. **Linear DAE** (Section 2.1):
$$E\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{f}(t)$$
2. **Semi-explicit DAE:**
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, t), \qquad \mathbf{0} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{y}, t)$$
3. **Quasi-linear DAE:**
$$E(\boldsymbol{x}, t)\boldsymbol{x}' = \boldsymbol{G}(\boldsymbol{x}, t)$$
4. **Constrained mechanics:**
$$\boldsymbol{M}\boldsymbol{v}' - \boldsymbol{\phi}_q^T\boldsymbol{\lambda} = \boldsymbol{n}, \qquad \boldsymbol{\phi}(\boldsymbol{q}, t) = \mathbf{0}$$
5. **Consistent initial values:** all invariants $\boldsymbol{\Psi}_j = \mathbf{0}$ (original and hidden constraints, e.g. $\boldsymbol{G}_{a,j}$ of every pass) hold at t_0:
$$\boldsymbol{\Psi}_j(\boldsymbol{x}_0, t_0) = \mathbf{0}$$

2.8.1 Master problem

$$E(x,t)x' = G(x,t), \qquad x(t_0) = x_0 \text{ consistent}$$

2.8.2 Index tests

DAE	index	condition
$F_{x'}$ nonsingular, or E nonsingular	0	none
linear, regular pencil	ν	$N^\nu = 0 \neq N^{\nu-1}$
semi-explicit	1	g_y nonsingular
semi-explicit, $0 = g(x,t)$	2	$g_x f_y$ nonsingular
$x' = f(x,y,z,t)$, $y' = g(x,y,t)$, $0 = h(y,t)$	3	$h_y g_x f_z$ nonsingular
Hessenberg of size r	r	$g_{x_1} f_{1,x_2} \cdots f_{r-1,x_r}$ nonsingular
holonomic mechanics	3	M positive definite, ϕ_q full row rank
dummy-variable algorithm	differentiations +1	B nonsingular at the last pass

2.8.3 Index reduction

Given $E(x,t)x' = G(x,t)$ with rank $E < n$, repeat

$$\begin{cases} E_d x' = G_d & \text{differential rows} \\ 0 = G_a(x,t) & \text{algebraic rows} \\ G_a^t + G_a^x x' = 0 & \text{differentiate, stack under differential rows} \end{cases}$$

until $\widetilde{E}$ is invertible; the number of differentiations is the index. The result is

$$\begin{cases} \widetilde{E}x' = \widetilde{G} & \text{underlying ODE} \\ G_{a,j}(x,t) = 0, \quad j = 1,\dots,\text{index} & \text{invariants (algebraic rows of pass } j\text{), to hold at } (x_0,t_0) \end{cases}$$

For the linear DAE $Ex' + Ax = f$: $G = f - Ax$, so $\widetilde{E} = (E_d; A_a)$, $\widetilde{A} = (A_d; 0)$, $\widetilde{f} = (f_d; f_a')$.

2.8.4 Dummy-variable loop

For $x_d' = f + Ax_a$, $0 = g + Bx_a$, repeat until $\widetilde{B}$ is nonsingular (index = passes +1):

$$\begin{cases} B_d x_a = -g_d & \text{part with algebraic variables} \\ \dfrac{\partial g_a}{\partial x_d} A\, x_a = -\left(\dfrac{\partial g_a}{\partial t} + \dfrac{\partial g_a}{\partial x_d} f \right) & \text{differentiated part free of algebraic variables, stacked below} \end{cases}$$

2.8.5 Constrained mechanics

$$
\begin{cases}
\boldsymbol{q}' = \boldsymbol{v} & \text{kinematics} \\
\boldsymbol{M}\boldsymbol{v}' - \boldsymbol{\phi}_q^T \boldsymbol{\lambda} = \boldsymbol{n} & \text{dynamics} \\
\boldsymbol{\phi}(\boldsymbol{q}, t) = \boldsymbol{0} & \text{position constraint} \\
\boldsymbol{\phi}_q \boldsymbol{v} + \boldsymbol{\phi}_t = \boldsymbol{0} & \text{velocity constraint (hidden)} \\
\boldsymbol{\phi}_q \boldsymbol{v}' + \boldsymbol{m} = \boldsymbol{0} & \text{acceleration constraint (replaces position)} \\
\boldsymbol{\lambda} = -\boldsymbol{D}^{-1} \left(\boldsymbol{m} + \boldsymbol{\phi}_q \boldsymbol{M}^{-1} \boldsymbol{n} \right) & \text{multiplier} \\
\ddot{\boldsymbol{\phi}} + 2\eta\omega\dot{\boldsymbol{\phi}} + \omega^2 \boldsymbol{\phi} = \boldsymbol{0} & \text{Baumgarte (against drift)}
\end{cases}
$$

with $\boldsymbol{D} = \boldsymbol{\phi}_q \boldsymbol{M}^{-1} \boldsymbol{\phi}_q^T$, $\boldsymbol{m}$ the terms of $\ddot{\boldsymbol{\phi}}$ without $\boldsymbol{v}'$, $\boldsymbol{n} = \boldsymbol{\tau} - \boldsymbol{C}\boldsymbol{v} - \boldsymbol{g}$.

3. Calculus of Variations

Everything needed to set up and solve functional-minimization exercises: first variation, Euler–Lagrange and Euler–Poisson equations, boundary terms, second-order tests, and integral and path constraints. Colors: integrand L, endpoint cost Φ, dynamics, equality constraints, inequality constraints, boundary conditions, variations, labels.

3.1 Variational Foundations

First and second variation. For $x, \eta \in \mathbb{V}$ and $g(\alpha) = \mathcal{J}(x + \alpha\eta)$:

$$\delta\mathcal{J}(x;\eta) = g'(0) = \lim_{\alpha\to 0} \frac{\mathcal{J}(x+\alpha\eta) - \mathcal{J}(x)}{\alpha}, \qquad \delta^2\mathcal{J}(x;\eta) = g''(0).$$

Stationarity (necessary condition). An η is **admissible** if $x + \alpha\eta$ stays admissible for all small $|\alpha|$. If $x^\star$ is an interior local minimizer, then

$$\delta\mathcal{J}(x^\star;\eta) = 0, \qquad \delta^2\mathcal{J}(x^\star;\eta) \geq 0 \qquad \text{for every admissible } \eta.$$

Integral functional. Differentiation under the integral sign is allowed when L is C^1 and x, η are C^1:

$$\mathcal{J}(x) = \int_a^b L(x, x', t)\,\mathrm{d}t, \qquad \delta\mathcal{J}(x;\eta) = \int_a^b \left(L_x\,\eta + L_{x'}\,\eta'\right)\mathrm{d}t.$$

Watch out.

- Stationary does not mean minimizer (maximum, saddle). If L is jointly convex in (x, x'), a stationary admissible function is a global minimizer (strict convexity: unique).
- The admissible class is part of the problem: $\int_{-1}^{1}(x-|t|)^2\mathrm{d}t$ has minimizer $|t|$ in $C([-1,1])$, but in $C^1([-1,1])$ the infimum 0 is not attained.

3.2 Fundamental Lemma and first variation

Fundamental lemma. If $f \in C([a,b])$ and $\int_a^b f\,g\,\mathrm{d}t = 0$ for every $g \in C_c^\infty$ with support in (a,b), then $f \equiv 0$ on $[a,b]$.

Independent functions. For $\int_a^b L(x,y,t)\,\mathrm{d}t$ with x, y independent:

$$\delta\mathcal{F} = \int_a^b \big(L_x\,\delta x + L_y\,\delta y\big)\,\mathrm{d}t,$$

linear in $(\delta x, \delta y)$; set one of them to zero to isolate each coefficient.

Recipe (stationarity with boundary terms). For $\mathcal{F}(x) = \int_a^b G(x,x',t)\,\mathrm{d}t + x(a)\,x(b)$:

1. Perturb $x \to x + \alpha\,\delta x$, differentiate at $\alpha = 0$: integral part plus boundary part.
2. Integrate by parts: $\int_a^b G_{x'}\,\delta x'\,\mathrm{d}t = \big[G_{x'}\,\delta x\big]_a^b - \int_a^b \big(\frac{\mathrm{d}}{\mathrm{d}t}G_{x'}\big)\,\delta x\,\mathrm{d}t$.
3. Collect the coefficients of δx in (a,b), of $\delta x(b)$ and of $\delta x(a)$.
4. Fundamental lemma on the coefficient of δx in (a,b); at a **free** endpoint its coefficient vanishes; at a **prescribed** endpoint $\delta x = 0$ and no condition arises.

$$\delta\mathcal{F} = \int_a^b \big(G_x\,\delta x + G_{x'}\,\delta x'\big)\mathrm{d}t + x(b)\,\delta x(a) + x(a)\,\delta x(b)$$

integration by parts

$$= \int_a^b \Big[G_x - \frac{\mathrm{d}}{\mathrm{d}t}G_{x'}\Big]\delta x\,\mathrm{d}t + \big(x(b) - G_{x'}(a)\big)\delta x(a) + \big(x(a) + G_{x'}(b)\big)\delta x(b)$$

$$\begin{cases} G_x - \dfrac{\mathrm{d}}{\mathrm{d}t}G_{x'} = 0 & \text{Euler–Lagrange} \\ x(a) + G_{x'}\big(x(b), x'(b), b\big) = 0 & \text{coefficient of } \delta x(b) \\ x(b) - G_{x'}\big(x(a), x'(a), a\big) = 0 & \text{coefficient of } \delta x(a) \end{cases}$$

Watch out.

- The lemma applies to the coefficient of δx, never to a coefficient of $\delta x'$: integrate by parts first. If $\delta\mathcal{F} = \int_a^b \psi\,\delta x'\,\mathrm{d}t = 0$ for all admissible δx, integration by parts gives $\psi' = 0$ (so $\psi = C$) and $\psi = 0$ at free ends, not $\psi = 0$ by the lemma applied to ψ.

3.3 Euler–Lagrange Equations

Problem.

$$\begin{aligned} &\text{minimize} \quad \mathcal{A}(x) = \int_a^b L(x,x',t)\,\mathrm{d}t \\ &\text{in} \quad x \in C^2([a,b]),\ x(a) = x_a,\ x(b) = x_b, \qquad \delta x \in C^\infty,\ \delta x(a) = \delta x(b) = 0. \end{aligned}$$

Conditions.

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}L_{x'} - L_x = 0 & \text{Euler–Lagrange (equivalently } L_x - \frac{\mathrm{d}}{\mathrm{d}t}L_{x'} = 0) \\ x(a) = x_a & \text{boundary condition} \\ x(b) = x_b & \text{boundary condition} \end{cases}$$

Expanded form: $\frac{\mathrm{d}}{\mathrm{d}t}L_{x'} = L_{x'x'}\,x'' + L_{x'x}\,x' + L_{x't}$.

Recipe. Compute L_x and $L_{x'}$; differentiate $L_{x'}$ along $x(t)$ (chain rule); impose the boundary data.

Watch out.

- L independent of x: $L_{x'} = C$ (first integral).

3.3.1 Beltrami identity

$L = L(x, x')$ of class C^2 with no explicit t, $x \in C^2$ solving Euler–Lagrange:

$$L - x'\,L_{x'} = C \qquad \text{(constant on } [a, b]\text{)}.$$

Watch out.

- $\frac{\mathrm{d}}{\mathrm{d}t}\big(L - x'L_{x'}\big) = L_t - x'\big(\frac{\mathrm{d}}{\mathrm{d}t}L_{x'} - L_x\big)$: the identity implies Euler–Lagrange only **where** $x' \neq 0$. Arcs with $x' = 0$ satisfy it automatically and must be checked in Euler–Lagrange.
- With explicit t the quantity $L - x'L_{x'}$ is not constant: its derivative is L_t.

3.3.2 Corner conditions (Weierstrass–Erdmann)

Broken extremal: x continuous, C^1 on $[a, c]$ and on $[c, b]$, corner at $c \in (a, b)$ with $p^{\pm} = x'(c^{\pm})$, $p^- \neq p^+$; $[\,G\,]_{c^-}^{c^+} = G^+ - G^-$ with $G^{\pm}$ evaluated at $(x(c), p^{\pm}, c)$. A minimizer satisfies

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}L_{x'} - L_x = 0 \ \text{ on } (a, c) \text{ and } (c, b) & \text{Euler–Lagrange on both arcs} \\ \Big[L_{x'}\Big]_{c^-}^{c^+} = 0 & \text{WE1: continuity of } L_{x'} \\ \Big[L - x'L_{x'}\Big]_{c^-}^{c^+} = 0 & \text{WE2} \end{cases}$$

WE2 holds also when L depends on t (derivation assumes the left arc extends smoothly slightly beyond c). For $\boldsymbol{x} \in \mathbb{R}^n$: WE1 componentwise, WE2 with $\boldsymbol{x}' \cdot L_{x'}$.

Recipe. Solve Euler–Lagrange on $[a, c]$ and on $[c, b]$ (unknowns: the four integration constants and c); impose continuity of x at c, WE1, WE2 and the boundary conditions.

Watch out.

- $L_{x'x'} > 0$ (strict convexity in x'): no corners.
- WE1 and WE2 together give $E\big(x(c), p^-, p^+, c\big) = 0$ (E the excess function below): a corner needs a bitangent of $p \mapsto L(x, p, t)$.
- A corner can win: for $\int_0^2 (x'^2 - 1)^2 \mathrm{d}t$, $x(0) = x(2) = 0$, the tent with slopes ± 1 has $\mathcal{F} = 0$, the smooth extremal $x \equiv 0$ has $\mathcal{F} = 2$.

3.4 Second Variation

Weak and strong minimizers. Setting: $\mathcal{F}(x) = \int_a^b L(x, x', t)\,\mathrm{d}t$ on $\mathcal{A} = \{x \in C^1([a, b]) \mid x(a) = x_a,\ x(b) = x_b\}$, with

$$\|x\|_0 = \max_{[a,b]} |x|, \qquad \|x\|_1 = \max_{[a,b]} |x| + \max_{[a,b]} |x'|.$$

- **weak** local minimizer: there is $\varepsilon > 0$ with $\mathcal{F}(x^\star) \leq \mathcal{F}(x)$ for all $x \in \mathcal{A}$ with $\|x - x^\star\|_1 < \varepsilon$;
- **strong** local minimizer: the same with $\|x - x^\star\|_0 < \varepsilon$.

Strict: strict inequality for $x \neq x^\star$. Strong $\Rightarrow$ weak, not conversely. Same definitions for continuous piecewise C^1 functions.

Second variation at an extremal $x^\star$, for $\eta(a) = \eta(b) = 0$, derivatives of L along $(x^\star, x^{\star\prime}, t)$:

$$\mathcal{F}(x^\star + \alpha\eta) = \mathcal{F}(x^\star) + \tfrac{\alpha^2}{2}\,\delta^2\mathcal{F}(x^\star;\eta) + o(\alpha^2),$$
$$\delta^2\mathcal{F}(x^\star;\eta) = \int_a^b \left[L_{xx}\,\eta^2 + 2L_{xx'}\,\eta\eta' + L_{x'x'}\,(\eta')^2\right]\mathrm{d}t = \int_a^b \left[P\,(\eta')^2 + Q\,\eta^2\right]\mathrm{d}t,$$
$$P = L_{x'x'}, \qquad Q = L_{xx} - \frac{\mathrm{d}}{\mathrm{d}t}L_{xx'}.$$

Weak local minimizer $\Rightarrow \delta^2\mathcal{F} \geq 0$ for all admissible η; $\delta^2\mathcal{F} < 0$ for some $\eta \Rightarrow$ not a minimizer.

3.4.1 Legendre condition

$x^\star$ weak local minimizer $\Rightarrow P(t) = L_{x'x'}\big(x^\star, x^{\star\prime}, t\big) \geq 0$ on $[a, b]$. **Strengthened Legendre:** $P > 0$ on $[a, b]$.

Watch out.

- Necessary, not sufficient: with $P > 0$ the term $Q\,\eta^2$ can still win on a long interval (see Jacobi).

3.4.2 Jacobi equation and conjugate points

$$-\frac{\mathrm{d}}{\mathrm{d}t}\big(P(t)\,v'(t)\big) + Q(t)\,v(t) = 0$$

$c \in (a, b]$ is **conjugate** to a if a solution $v \not\equiv 0$ has $v(a) = 0$ and $v(c) = 0$.

Recipe. For $P > 0$: solve Jacobi with $v(a) = 0$, $v'(a) = 1$; the conjugate points are the zeros of v in $(a, b]$.

Jacobi condition ($x^\star$ extremal, $P > 0$ on $[a, b]$):

- necessary: $x^\star$ weak local minimizer $\Rightarrow$ no conjugate point in (a, b);
- sufficient: no conjugate point in $(a, b] \Rightarrow x^\star$ strict weak local minimizer.

Watch out.

- $\frac{1}{2}\int_0^{\,b}(x'^2 - x^2)\mathrm{d}t$, $x(0) = 0$, $x(b) = \beta$: $P = 1$, $Q = -1$, $v = \sin t$, conjugate point π; not a minimizer if $b > \pi$, strict weak minimizer if $b < \pi$.

3.4.3 Weierstrass excess function and condition

$$E(x, p, q, t) = L(x, q, t) - L(x, p, t) - (q - p)\,L_{x'}(x, p, t).$$

$L \in C^2$, $x^\star \in C^2$ extremal: $x^\star$ strong local minimizer $\Rightarrow$

$$E\big(x^\star(t), x^{\star\prime}(t), q, t\big) \geq 0 \qquad \forall t \in [a, b],\ \forall q \in \mathbb{R}.$$

Watch out.

- $E = \frac{1}{2}L_{x'x'}(\xi)(q - p)^2$ for some ξ between p and q: Weierstrass $\Rightarrow$ Legendre. L convex in $x' \Rightarrow E \geq 0$ automatically.
- $\int_0^{\,1} x'^3\mathrm{d}t$, $x(0) = 0$, $x(1) = 1$: $x^\star = t$ has $P = 6 > 0$, no conjugate point (weak minimizer), but $E(1, q) = (q - 1)^2(q + 2) < 0$ for $q < -2$: not a strong minimizer.

3.4.4 Classification of an extremal

Recipe.

1. Compute $P = L_{x'x'}$ and Q along $x^\star$; $P < 0$ somewhere: not a weak minimizer.
2. If $P > 0$ on $[a, b]$: solve Jacobi with $v(a) = 0$, $v'(a) = 1$. A zero in (a, b): not a weak minimizer; no zero in $(a, b]$: strict weak minimizer (a first zero exactly at b is inconclusive).
3. Strong minimum: check $E(x^\star, x^{\star\prime}, q, t) \geq 0$ for all q (automatic if L is convex in x'); failure means not a strong minimizer.

3.4.5 Table of second-order tests

Condition	Type	Statement along $x^\star$
Euler–Lagrange	necessary (first order)	$\frac{\mathrm{d}}{\mathrm{d}t}L_{x'} - L_x = 0$
Legendre	necessary for a weak minimum	$L_{x'x'} \geq 0$
Jacobi	necessary for a weak minimum (with $L_{x'x'} > 0$)	no conjugate point in (a, b)
Legendre + Jacobi	sufficient for a strict weak minimum	$L_{x'x'} > 0$, no conjugate point in $(a, b]$
Weierstrass	necessary for a strong minimum	$E(x^\star, x^{\star\prime}, q, t) \geq 0$ for all q

3.5 Higher-Order Functionals

Problem.

$$\begin{aligned} \text{minimize} \quad & \mathcal{B}(x) = \int_a^b L\left(x, x', x'', t\right) \mathrm{d}t \\ \text{in} \quad & x \in C^4, \\ & x(a),\ x'(a),\ x(b),\ x'(b) \text{ prescribed}, \\ & \delta x = \delta x' = 0 \text{ at } a \text{ and } b. \end{aligned}$$

Conditions.

$$\begin{cases} L_x - \dfrac{\mathrm{d}}{\mathrm{d}t}L_{x'} + \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}L_{x''} = 0 & \text{Euler–Poisson} \\ x(a),\ x'(a),\ x(b),\ x'(b) \text{ prescribed} & \text{four boundary data} \end{cases}$$

Order m, $L\left(t, x, x', \dots, x^{(m)}\right)$:

$$\sum_{k=0}^{m} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(L_{x^{(k)}}\right) = 0 \qquad \text{(Euler–Poisson, order } m\text{)}$$

The equation has order $2m$ when $L_{x^{(m)}x^{(m)}} \neq 0$ and needs $2m$ boundary data.

Natural boundary conditions. If some data are free, keep the boundary terms:

$$\delta\mathcal{B} = \int_a^b (A)\,\delta x\,\mathrm{d}t + (B)\,\delta x(b) - (C)\,\delta x(a) + (D)\,\delta x'(b) - (E)\,\delta x'(a),$$

$$(A) = L_x - \frac{\mathrm{d}}{\mathrm{d}t}L_{x'} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}L_{x''},$$

$$(B) = \left(L_{x'} - \frac{\mathrm{d}}{\mathrm{d}t}L_{x''}\right)\Big|_b, \qquad (C) = \left(L_{x'} - \frac{\mathrm{d}}{\mathrm{d}t}L_{x''}\right)\Big|_a,$$

$$(D) = L_{x''}\Big|_b, \qquad (E) = L_{x''}\Big|_a.$$

$(A) = 0$ always; each **free** datum gives its coefficient $= 0$, each **prescribed** datum gives nothing:

Datum	Prescribed	Free: natural condition
$x(b)$	$\delta x(b) = 0$	$(B) = \left(L_{x'} - \frac{\mathrm{d}}{\mathrm{d}t}L_{x''}\right)\Big\|_b = 0$
$x(a)$	$\delta x(a) = 0$	$(C) = \left(L_{x'} - \frac{\mathrm{d}}{\mathrm{d}t}L_{x''}\right)\Big\|_a = 0$
$x'(b)$	$\delta x'(b) = 0$	$(D) = L_{x''}\Big\|_b = 0$
$x'(a)$	$\delta x'(a) = 0$	$(E) = L_{x''}\Big\|_a = 0$

Watch out.

- Each free datum gives exactly one natural condition from the table: $L = (x'')^2 - x^2 + t$ gives $x'''' - x = 0$; with only $x(a) = 0$, $x'(b) = 2$ prescribed: $x'''(b) = 0$, $x''(a) = 0$, $x(a) = 0$, $x'(b) = 2$.
- The necessary conditions may be inconsistent (no stationary point): $\int_0^1 \left(\frac{1}{2}y'^2 + yy' + y\right)\mathrm{d}x$ with only $y(1) = 1$ gives $y'' = 1$, $y'(0) + y(0) = 0$, $y(1) = 1$, which has no solution.

3.6 Boundary Conditions

3.6.1 Endpoint constraints with boundary multipliers

Problem. Handle $w_k = 0$ with multipliers.

$$\begin{aligned} &\text{minimize} && \int_a^b L(z, z', t)\,\mathrm{d}t \\ &\text{subject to} && w_k\big(z(a), z(b)\big) = 0, \quad k = 1, \dots, p \end{aligned}$$

Recipe. Lagrangian $\displaystyle\int_a^b L\,\mathrm{d}t - \sum_{k=1}^{p} \lambda_k\, w_k$; require $\delta = 0$ for δz and $\delta\lambda_k$ in turn.

Conditions. Unknowns: $z(t)$ and $\lambda_1, \dots, \lambda_p$.

$$\begin{cases} L_z - \dfrac{\mathrm{d}}{\mathrm{d}t}L_{z'} = 0 & \text{Euler–Lagrange} \\ w_k\big(z(a), z(b)\big) = 0 & \text{constraints } (\delta\lambda_k) \\ L_{z'}\big(z(a), z'(a), a\big) + \displaystyle\sum_{k=1}^{p} \lambda_k \frac{\partial w_k}{\partial z(a)} = 0 & \text{boundary condition at } a \\ L_{z'}\big(z(b), z'(b), b\big) - \displaystyle\sum_{k=1}^{p} \lambda_k \frac{\partial w_k}{\partial z(b)} = 0 & \text{boundary condition at } b \end{cases}$$

Fixed ends as a special case. $w_1 = x(a) - x_a$, $w_2 = x(b) - x_b$: the boundary conditions read $-\lambda_1 - L_{x'}(a) = 0$, $-\lambda_2 + L_{x'}(b) = 0$, always solvable for λ_1, λ_2 and without information; one is left with Euler–Lagrange and $x(a) = x_a$, $x(b) = x_b$.

3.6.2 Endpoint costs and free endpoints

Problem.

$$\mathcal{F}(x) = \int_a^b L(x, x', t)\,\mathrm{d}t + \Phi\big(x(a), x(b)\big)$$

$$\delta\mathcal{F} = \int_a^b \left[L_x - \frac{\mathrm{d}}{\mathrm{d}t}L_{x'}\right]\delta x\,\mathrm{d}t + \Big(\Phi_{x_a} - L_{x'}\big(x(a), x'(a), a\big)\Big)\delta x(a) + \Big(\Phi_{x_b} + L_{x'}\big(x(b), x'(b), b\big)\Big)\delta x(b)$$

Φ_{x_a}, Φ_{x_b}: partial derivatives of Φ with respect to its first and second argument. At a free endpoint δx is arbitrary and its coefficient vanishes:

Boundary data at an endpoint	Condition to impose
$x(a) = x_a$ or $x(b) = x_b$ prescribed	$\delta x = 0$ there: no extra condition
$x(a)$ free, no endpoint cost	$L_{x'}\big(x(a), x'(a), a\big) = 0$
$x(b)$ free, no endpoint cost	$L_{x'}\big(x(b), x'(b), b\big) = 0$
$x(a)$ free, endpoint cost Φ	$\Phi_{x_a} - L_{x'}\big(x(a), x'(a), a\big) = 0$
$x(b)$ free, endpoint cost Φ	$\Phi_{x_b} + L_{x'}\big(x(b), x'(b), b\big) = 0$
$w\big(x(a), x(b)\big) = 0$ coupling the ends	$w = 0$ and the multiplier conditions above

Watch out.

- Free ends and L independent of x: $L_{x'} = C$ and $L_{x'} = 0$ at both ends give $C = 0$ (e.g. $\int_0^1 x'(\log x')^2 \mathrm{d}t$: $\log x'\,(\log x' + 2) = 0$).
- Mind the sign at a versus b: $x(0) + \int_0^1 (tx^2 + x'^2)\mathrm{d}t$, no boundary data: $tx - x'' = 0$, $\Phi_{x_a} - L_{x'}(0) = 1 - 2x'(0) = 0$, $x'(1) = 0$.

3.7 Isoperimetric Constraints

Problem.

$$\text{minimize} \quad \mathcal{F}(x) = \int_a^b F\big(x, x', \dots, x^{(n)}, t\big)\,\mathrm{d}t$$

$$\text{subject to} \quad \mathcal{G}_k(x) = \int_a^b G_k\big(x, x', \dots, x^{(n)}, t\big)\,\mathrm{d}t = 0, \quad k = 1, \dots, p$$

Multiplier rule (normal case). Stationary candidates are stationary points, with respect to $x(t)$ **and** the constants λ_k (one per constraint), of

$$\mathcal{L}(x, \boldsymbol{\lambda}) = \mathcal{F}(x) - \sum_{k=1}^{p} \lambda_k\,\mathcal{G}_k(x) = \int_a^b \Big(F - \sum_{k=1}^{p} \lambda_k G_k\Big)\mathrm{d}t.$$

Recipe.

1. Form the augmented integrand $F - \sum_k \lambda_k G_k$ with constant λ_k.
2. Write its Euler–Lagrange (Euler–Poisson for higher order) equation.

3. Add the boundary conditions, prescribed or natural, computed with the augmented integrand.
4. Solve with the λ_k as parameters, then impose $\mathcal{G}_k = 0$ (from $\delta\lambda_k$) to fix them.
5. If the constraint qualification fails, see the abnormal case.

General problem.

$$\begin{aligned} \text{minimize} \quad & \int_a^b L(x, x', t)\,\mathrm{d}t + \Phi\big(x(a), x(b)\big) \\ \text{subject to} \quad & w_k\big(x(a), x(b)\big) = 0, \quad k = 1, \dots, p \\ & \int_a^b g_k(x, x', t)\,\mathrm{d}t + \Psi_k\big(x(a), x(b)\big) = 0, \quad k = 1, \dots, m \end{aligned}$$

$$M(x, x', \boldsymbol{\lambda}, t) = L(x, x', t) - \sum_{k=1}^{m} \lambda_k\, g_k(x, x', t),$$

$$N(x_a, x_b, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \Phi(x_a, x_b) - \sum_{k=1}^{m} \lambda_k\, \Psi_k(x_a, x_b) - \sum_{k=1}^{p} \mu_k\, w_k(x_a, x_b).$$

Conditions (normal case).

$$\begin{cases} M_x - \dfrac{\mathrm{d}}{\mathrm{d}t} M_{x'} = 0 & \text{Euler–Lagrange} \\ N_{x_a}\big(x(a), x(b), \boldsymbol{\lambda}, \boldsymbol{\mu}\big) - M_{x'}\big(x(a), x'(a), \boldsymbol{\lambda}, a\big) = 0 & \text{adjoint BC at } a \\ N_{x_b}\big(x(a), x(b), \boldsymbol{\lambda}, \boldsymbol{\mu}\big) + M_{x'}\big(x(b), x'(b), \boldsymbol{\lambda}, b\big) = 0 & \text{adjoint BC at } b \\ w_k\big(x(a), x(b)\big) = 0 & \text{endpoint constraints } (\delta\mu_k),\ k = 1, \dots, p \\ \displaystyle\int_a^b g_k\,\mathrm{d}t + \Psi_k\big(x(a), x(b)\big) = 0 & \text{integral constraints } (\delta\lambda_k),\ k = 1, \dots, m \end{cases}$$

For prescribed boundary values the corresponding endpoint equation is solved by the μ_k and drops out.

Abnormal case. The normal case needs a constraint qualification (first variations of the constraints linearly independent). If it fails, use $M = \lambda_0 L + \sum_k \lambda_k g_k$ with $\lambda_0 \geq 0$, $(\lambda_0, \boldsymbol{\lambda}) \neq 0$; the normal case is $\lambda_0 = 1$ (note the + sign: the sign of a multiplier is a convention).

Watch out.

- Do not apply the fundamental lemma to the constrained variations ($\int_0^1 \delta x\,\mathrm{d}t = 0$ for $\int_0^1 x\,\mathrm{d}t = 1$): use a multiplier.
- λ is a constant, fixed by the constraint: min $\int_0^1 x'^2\mathrm{d}t$, $\int_0^1 x\,\mathrm{d}t = 1$, $x(0) = 2$, $x(1) = 1$: $2x'' + \lambda = 0$ and the constraint give $\lambda = -12$, $x = 3t^2 - 4t + 2$ (global minimizer by convexity).

3.8 Path Constraints

Problem (equality path constraint).

$$\begin{aligned} \text{minimize} \quad & \int_a^b L(x, x', t)\,\mathrm{d}t \\ \text{subject to} \quad & g(x, x', t) = 0 \quad \forall t \in [a, b] \end{aligned}$$

$$M(x, x', \lambda, t) = L(x, x', t) - \lambda(t)\, g(x, x', t).$$

The multiplier is a **function** $\lambda(t)$.

Conditions (equality). Free endpoints, unknowns $\big(x(t), \lambda(t)\big)$:

$$\begin{cases} M_x - \dfrac{\mathrm{d}}{\mathrm{d}t} M_{x'} = 0 & \text{Euler–Lagrange of } M \\ M_{x'}\big(x(b), x'(b), \lambda(b), b\big) = 0 & \text{natural BC at } b \\ -M_{x'}\big(x(a), x'(a), \lambda(a), a\big) = 0 & \text{natural BC at } a \\ -M_\lambda = g(x, x', t) = 0 & \text{constraint} \end{cases}$$

A prescribed endpoint value replaces the natural condition at that end. Euler–Lagrange plus $g = 0$ is a differential–algebraic system.

Problem (inequality path constraint).

$$\begin{aligned} &\text{minimize} && \int_a^b L(x, x', t)\,\mathrm{d}t \\ &\text{subject to} && c(x, x', t) \le 0 \quad \forall t \in [a, b] \end{aligned} \qquad M(x, x', \mu, t) = L(x, x', t) + \mu(t)\, c(x, x', t).$$

Conditions (inequality). Under a constraint qualification:

$$\begin{cases} M_x - \dfrac{\mathrm{d}}{\mathrm{d}t} M_{x'} = 0 & \text{stationarity} \\ c(x, x', t) \le 0 & \text{primal feasibility} \\ \mu(t) \ge 0 & \text{dual feasibility} \\ \mu(t)\, c(x, x', t) = 0 & \text{complementarity} \end{cases}$$

$\mu = 0$ on inactive arcs; $\mu > 0$ only where the constraint is active.

Recipe.

1. Form M: $L - \lambda g$ for an equality, $L + \mu c$ for an inequality.
2. Write Euler–Lagrange of M together with the constraint, with the boundary conditions (prescribed, or natural as above).
3. Inequality: guess the active and inactive arcs; set $\mu = 0$ on inactive arcs and $c = 0$ on active ones.
4. Check $c \le 0$ and $\mu \ge 0$, and impose the Weierstrass–Erdmann conditions for M at the junctions.

Watch out.

- Signs: equality $M = L - \lambda g$, inequality $M = L + \mu c$ with $\mu \ge 0$.
- Regularity: $\lambda(t)$, $\mu(t)$ are not automatically regular (for state inequalities and changing active sets the multiplier can even be a measure); the pointwise equations hold under smoothness and constraint-qualification hypotheses.
- Junctions: at active/inactive junctions also impose the Weierstrass–Erdmann matching conditions for the augmented integrand M; solving the ODE separately on each arc does not give them.

3.9 Summary

1. **Notation**

$$\left.\frac{\mathrm{d}}{\mathrm{d}\alpha}\right|_{\alpha=0} = \delta$$

2. **Problem: minimize**

$$\int_a^b L(x, x', t)\,\mathrm{d}t \;+\; \Phi(x(a), x(b))$$

3. **Boundary conditions:**

$$x(a) = x_a, \quad x(b) = \text{free}, \quad w_k(x(a), x(b)) = 0, \quad \cdots$$

4. **Integral constraints:**

$$\int_a^b g_k(x, x', t)\,\mathrm{d}t \;+\; \Psi_k(x(a), x(b)) = 0$$

5. **Higher order problem: minimize**

$$\int_a^b L(x, x', x'', \ldots, x^{(n)}, t)\,\mathrm{d}t \;+\; \Phi(x(a), x(b))$$

3.9.1 Admissible Spaces

- $\mathbb{V} = \left\{ x \in C^m([a,b]), \ldots \right\}$
- $\mathbb{D} = \left\{ \delta x \in C^\infty([a,b]), \delta x(a) = \delta x(b) = 0, \delta x'(a) = \delta x'(b) = 0, \ldots \right\}$

3.9.1.1 Endpoint Costs

$$\begin{aligned}
\mathcal{F}(x) &= \int_a^b L(x, x', t)\,\mathrm{d}t + \Phi(x(a), x(b)) \\
\delta\mathcal{F}(x) &= \delta \int_a^b L(x, x', t)\,\mathrm{d}t + \delta\Phi(x(a), x(b)) \\
&= \int_a^b \delta L(x, x', t)\,\mathrm{d}t + \delta\Phi(x(a), x(b)) \\
&= \int_a^b L_x(x, x', t)\delta x + L_{x'}(x, x', t)\delta x'\,\mathrm{d}t \\
&\quad + \Phi_{x_a}(x(a), x(b))\delta x(a) + \Phi_{x_b}(x(a), x(b))\delta x(b)
\end{aligned}$$

integration by parts

$$\begin{aligned}
&= \int_a^b \left[L_x(x, x', t) - \frac{\mathrm{d}}{\mathrm{d}t} L_{x'}(x, x', t) \right] \delta x\,\mathrm{d}t \\
&\quad + \left(\Phi_{x_a}(x(a), x(b)) - L_{x'}(x(a), x'(a), a) \right) \delta x(a) \\
&\quad + \left(\Phi_{x_b}(x(a), x(b)) + L_{x'}(x(b), x'(b), b) \right) \delta x(b)
\end{aligned}$$

3.9.1.2 Endpoint Constraints

Define

$$\begin{aligned}
\mathcal{F}(x) &= \int_a^b L(x, x', t)\,\mathrm{d}t - \sum_{k=1}^{p} \lambda_k w_k(x(a), x(b)) \\
\delta\mathcal{F}(x) &= \delta \int_a^b L(x, x', t)\,\mathrm{d}t - \delta \sum_{k=1}^{p} \lambda_k w_k(x(a), x(b)) \\
&= \int_a^b L_x(x, x', t)\delta x + L_{x'}(x, x', t)\delta x'\,\mathrm{d}t - \sum_{k=1}^{p} w_k(x(a), x(b))\delta\lambda_k \\
&\quad - \sum_{k=1}^{p} \lambda_k \left(\frac{\partial w_k}{\partial x_a}(x(a), x(b))\delta x(a) + \frac{\partial w_k}{\partial x_b}(x(a), x(b))\delta x(b) \right)
\end{aligned}$$

integration by parts

$$\begin{aligned}
&= \int_a^b \left[L_x(x, x', t) - \frac{\mathrm{d}}{\mathrm{d}t} L_{x'}(x, x', t) \right] \delta x\,\mathrm{d}t - \sum_{k=1}^{p} w_k(x(a), x(b))\delta\lambda_k \\
&\quad - \left(\sum_{k=1}^{p} \lambda_k \frac{\partial w_k}{\partial x_a}(x(a), x(b)) + L_{x'}(x(a), x'(a), a) \right) \delta x(a) \\
&\quad - \left(\sum_{k=1}^{p} \lambda_k \frac{\partial w_k}{\partial x_b}(x(a), x(b)) - L_{x'}(x(b), x'(b), b) \right) \delta x(b)
\end{aligned}$$

3.9.1.3 Integral Constraints

Define

$$\begin{aligned}
M(x, x', \boldsymbol{\lambda}, t) &= L(x, x', t) - \sum_{k=1}^{m} \lambda_k g_k(x, x', t) \\
N(x_a, x_b, \boldsymbol{\lambda}) &= \Phi(x_a, x_b) - \sum_{k=1}^{m} \lambda_k \Psi_k(x_a, x_b)
\end{aligned}$$

$$\mathcal{F}(x,\boldsymbol{\lambda}) = \int_a^b M(x,x',\boldsymbol{\lambda},t)\,\mathrm{d}t + N(x(a),x(b),\boldsymbol{\lambda})$$

$$\begin{aligned}\delta\mathcal{F}(x,\boldsymbol{\lambda}) &= \delta\int_a^b M(x,x',\boldsymbol{\lambda},t)\,\mathrm{d}t + \delta N(x(a),x(b),\boldsymbol{\lambda})\\ &= \int_a^b M_x(x,x',\boldsymbol{\lambda},t)\delta x + M_{x'}(x,x',\boldsymbol{\lambda},t)\delta x' + M_{\boldsymbol{\lambda}}(x,x',\boldsymbol{\lambda},t)\cdot\delta\boldsymbol{\lambda}\,\mathrm{d}t\\ &\quad + N_{x_a}(x(a),x(b),\boldsymbol{\lambda})\delta x(a) + N_{x_b}(x(a),x(b),\boldsymbol{\lambda})\delta x(b) + N_{\boldsymbol{\lambda}}\cdot\delta\boldsymbol{\lambda}\end{aligned}$$

integration by parts

$$\begin{aligned}&= \int_a^b \left[M_x(x,x',\boldsymbol{\lambda},t) - \frac{\mathrm{d}}{\mathrm{d}t}M_{x'}(x,x',\boldsymbol{\lambda},t)\right]\delta x\,\mathrm{d}t\\ &\quad - \sum_{k=1}^m \delta\lambda_k\left[\int_a^b g_k(x,x',t)\,\mathrm{d}t + \Psi_k(x(a),x(b))\right]\\ &\quad + \Big(N_{x_a}(x(a),x(b),\boldsymbol{\lambda}) - M_{x'}(x(a),x'(a),\boldsymbol{\lambda},a)\Big)\delta x(a)\\ &\quad + \Big(N_{x_b}(x(a),x(b),\boldsymbol{\lambda}) + M_{x'}(x(b),x'(b),\boldsymbol{\lambda},b)\Big)\delta x(b)\end{aligned}$$

3.9.1.4 Higher-Order Reduction

Transform to optimal control problem

$$\text{minimize:}\quad \int_a^b L(x_1,x_2,x_3,\dots,x_n,u,t)\,\mathrm{d}t$$

$$\text{subject to:}\quad \begin{cases} x_1' = x_2\\ x_2' = x_3\\ \quad\vdots\\ x_{n-1}' = x_n\\ x_n' = u\end{cases}$$

3.9.2 General Problem

$$\text{minimize:}\quad \int_a^b L(x,x',t)\,\mathrm{d}t \;+\; \Phi(x(a),x(b))$$

$$\text{subject to:}\quad w_k(x(a),x(b)) = 0, \qquad k = 1,2,\dots,p$$

$$\int_a^b g_k(x,x',t)\,\mathrm{d}t \;+\; \Psi_k(x(a),x(b)) = 0, \qquad k = 1,2,\dots,m$$

defining

$$M(x,x',\boldsymbol{\lambda},t) = L(x,x',t) - \sum_{k=1}^m \lambda_k g_k(x,x',t)$$

$$N(x_a,x_b,\boldsymbol{\lambda},\boldsymbol{\mu}) = \Phi(x_a,x_b) - \sum_{k=1}^m \lambda_k\Psi_k(x_a,x_b) - \sum_{k=1}^p \mu_k w_k(x_a,x_b)$$

the stationary points satisfy the BVP

$$
\begin{cases}
M_x(x, x', \boldsymbol{\lambda}, t) - \dfrac{\mathrm{d}}{\mathrm{d}t} M_{x'}(x, x', \boldsymbol{\lambda}, t) = 0 & \text{Euler–Lagrange} \\
N_{x_a}(x(a), x(b), \boldsymbol{\lambda}, \boldsymbol{\mu}) - M_{x'}(x(a), x'(a), \boldsymbol{\lambda}, a) = 0 & \text{Adjoint BC} \\
N_{x_b}(x(a), x(b), \boldsymbol{\lambda}, \boldsymbol{\mu}) + M_{x'}(x(b), x'(b), \boldsymbol{\lambda}, b) = 0 & \text{Adjoint BC} \\
w_k(x(a), x(b)) = 0 & \text{endpoint constraints, } k = 1, 2, \dots, p \\
\displaystyle\int_a^b g_k(x, x', t)\,\mathrm{d}t + \Psi_k(x(a), x(b)) = 0 & \text{integral constraints, } k = 1, 2, \dots, m
\end{cases}
$$

3.9.3 Path Constraints

Equality, with multiplier function $\lambda(t)$, and inequality, with multiplier function $\mu(t)$:

$$g(x, x', t) = 0, \quad M = L - \lambda(t)\, g$$

$$c(x, x', t) \le 0, \quad M = L + \mu(t)\, c, \quad \mu(t) \ge 0, \quad \mu(t)\, c(x, x', t) = 0$$

3.9.4 Minima and Existence

- Stationary is not minimal: joint convexity of L in (x, x') on a convex admissible set gives a global minimizer, strict convexity uniqueness.
- Classification: see the table of second-order tests.
- Existence by the direct method: nonempty closed admissible set, coercivity, compactness, lower semicontinuity (convexity in x' gives weak lower semicontinuity).

4. Optimal Control

Everything needed to set up and solve optimal control exercises: Hamiltonian and Pontryagin conditions, bang-bang and singular arcs, free final time, integral and path constraints, and the transformations to a standard form. Colors: running cost, endpoint cost, dynamics, equality/integral constraints, control set, inequalities, boundary conditions, variations, labels.

4.1 From Isoperimetric Problems to Optimal Control

Problem.

$$\begin{aligned} \text{minimize:} \quad & \int_a^b L(x, x', t)\,\mathrm{d}t \\ \text{subject to:} \quad & w\big(x(a), x(b)\big) = 0 \\ & \int_a^b \mathcal{G}(x, x', t)\,\mathrm{d}t = l \end{aligned}$$

Recipe.

1. Take $y = x'$ as control and x as state.
2. Add the state z with $z' = \mathcal{G}(x, y, t)$, $z(a) = 0$, $z(b) = l$.
3. Solve the optimal control problem

$$\begin{aligned} \text{minimize:} \quad & \int_a^b L(x, y, t)\,\mathrm{d}t \\ \text{subject to:} \quad & x' = y, \qquad z' = \mathcal{G}(x, y, t) \\ & w\big(x(a), x(b)\big) = 0 \\ & z(a) = 0, \qquad z(b) = l \end{aligned}$$

Watch out.

- The (constant) costate of z is the multiplier of the constraint. With $H = L + \eta\,\mathcal{G} + \ldots$ it equals $-\lambda$ of the calculus-of-variations convention $M = L - \lambda\mathcal{G}$.

4.2 Optimal Control Problems

4.2.1 Bolza form and first-order conditions

Problem ($\boldsymbol{x} \in \mathbb{R}^n$ state, $\boldsymbol{u} \in \mathbb{R}^m$ control, $\boldsymbol{b} \in \mathbb{R}^q$; pure Mayer: $L = 0$; pure Lagrange: $\Phi = 0$).

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{u}}{\text{minimize}} \quad & \Phi\big(\boldsymbol{x}(a), \boldsymbol{x}(b)\big) + \int_a^b L\big(\boldsymbol{x}, \boldsymbol{u}, t\big)\, \mathrm{d}t \\ \text{subject to} \quad & \boldsymbol{x}' = \boldsymbol{f}\big(\boldsymbol{x}, \boldsymbol{u}, t\big) \\ & \boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b)\big) = \boldsymbol{0} \\ & \boldsymbol{u}(t) \in \mathcal{U} \quad \text{a.e. on } [a, b] \end{aligned}$$

Hamiltonian and endpoint Lagrangian (normal minimization convention, used throughout).

$$\begin{aligned} H(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}, t) &= L(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{\lambda}^\top \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t), \\ B(\boldsymbol{x}_a, \boldsymbol{x}_b, \boldsymbol{\omega}) &= \Phi(\boldsymbol{x}_a, \boldsymbol{x}_b) + \boldsymbol{\omega}^\top \boldsymbol{b}(\boldsymbol{x}_a, \boldsymbol{x}_b). \end{aligned}$$

Recipe (variation of the augmented functional). $H_{\boldsymbol{x}}$, $H_{\boldsymbol{u}}$, $B_{\boldsymbol{x}_a}$, ... are column gradients.

$$\mathcal{J}_a = B\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{\omega}\big) + \int_a^b \Big[H(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{u}, t) - \boldsymbol{\lambda}^\top \boldsymbol{x}'\Big]\mathrm{d}t$$

$$\begin{aligned} \delta\mathcal{J}_a = {} & B_{\boldsymbol{x}_a}^\top \delta\boldsymbol{x}(a) + B_{\boldsymbol{x}_b}^\top \delta\boldsymbol{x}(b) + \boldsymbol{b}^\top \delta\boldsymbol{\omega} \\ & + \int_a^b H_{\boldsymbol{x}}^\top \delta\boldsymbol{x} + H_{\boldsymbol{u}}^\top \delta\boldsymbol{u} + \big(\boldsymbol{f} - \boldsymbol{x}'\big)^\top \delta\boldsymbol{\lambda} - \boldsymbol{\lambda}^\top \delta\boldsymbol{x}'\, \mathrm{d}t \end{aligned}$$

integration by parts

$$\begin{aligned} = {} & \int_a^b \big(H_{\boldsymbol{x}} + \boldsymbol{\lambda}'\big)^\top \delta\boldsymbol{x} + H_{\boldsymbol{u}}^\top \delta\boldsymbol{u} + \big(\boldsymbol{f} - \boldsymbol{x}'\big)^\top \delta\boldsymbol{\lambda}\, \mathrm{d}t \\ & + \big(B_{\boldsymbol{x}_a} + \boldsymbol{\lambda}(a)\big)^\top \delta\boldsymbol{x}(a) + \big(B_{\boldsymbol{x}_b} - \boldsymbol{\lambda}(b)\big)^\top \delta\boldsymbol{x}(b) + \boldsymbol{b}^\top \delta\boldsymbol{\omega} \end{aligned}$$

Setting to zero the coefficient of each variation gives the conditions.

Conditions (normal extremal; $H_{\boldsymbol{u}} = \boldsymbol{0}$ only for an **interior, unconstrained, smooth** control).

$$\begin{cases} \boldsymbol{x}' = H_{\boldsymbol{\lambda}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t) & \text{State equation} \\ \boldsymbol{\lambda}' = -H_{\boldsymbol{x}} & \text{Costate equation} \\ \boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b)\big) = \boldsymbol{0} & \text{Boundary conditions} \\ B_{\boldsymbol{x}_a} + \boldsymbol{\lambda}(a) = \boldsymbol{0} & \text{Costate BC at } a \\ B_{\boldsymbol{x}_b} - \boldsymbol{\lambda}(b) = \boldsymbol{0} & \text{Costate BC at } b \\ H_{\boldsymbol{u}} = \boldsymbol{0} & \text{Stationarity in } \boldsymbol{u} \end{cases}$$

Costate boundary conditions, component i of the state at an endpoint.

Component i of the state at an endpoint	Adjoint boundary condition
value prescribed by $\boldsymbol{b}$	the equation only defines the multiplier in $\boldsymbol{\omega}$: no information on λ_i

Component i of the state at an endpoint	Adjoint boundary condition
free, and Φ does not depend on it	$\lambda_i(a) = 0$ (initial) or $\lambda_i(b) = 0$ (final)
free, and Φ depends on it	$\lambda_i(a) = -\partial\Phi/\partial x_{a,i}$, $\lambda_i(b) = \partial\Phi/\partial x_{b,i}$
endpoints coupled, e.g. $\boldsymbol{x}(a) - \boldsymbol{x}(b) = \mathbf{0}$	eliminate $\boldsymbol{\omega}$ between the two equations (here $\boldsymbol{\lambda}(a) + \Phi_{x_a} = \boldsymbol{\lambda}(b) - \Phi_{x_b}$, i.e. $\boldsymbol{\lambda}(a) = \boldsymbol{\lambda}(b)$ if Φ does not depend on the endpoints)

Watch out.

- $H_{\boldsymbol{u}} = \mathbf{0}$ is only a first-order necessary condition: $H_{\boldsymbol{uu}} \geq 0$ is necessary, $H_{\boldsymbol{uu}} > 0$ gives a strict local minimizer of H in $\boldsymbol{u}$; convexity of H in $\boldsymbol{u}$ gives the global one.
- The BVP is a system of first-order ODEs (states, costates, and T, $\boldsymbol{p}$ as constant states if present): the number of conditions must equal the number of ODEs, after eliminating $\boldsymbol{u}$ with $H_{\boldsymbol{u}} = \mathbf{0}$ or the minimum condition.
- The components of $\boldsymbol{\omega}$ are eliminated, not computed.
- Maximize J = minimize $-J$: change the sign of L and Φ before forming H and B.

4.2.2 Pontryagin minimum principle

Conditions (normal case, usual regularity hypotheses; convexity of $\mathcal{U}$ not required).

$$\begin{cases} \boldsymbol{x}' = H_{\boldsymbol{\lambda}}, \qquad \boldsymbol{\lambda}' = -H_{\boldsymbol{x}} & \text{Canonical equations} \\ \boldsymbol{b} = \mathbf{0}, \quad B_{x_a} + \boldsymbol{\lambda}(a) = \mathbf{0}, \quad B_{x_b} - \boldsymbol{\lambda}(b) = \mathbf{0} & \text{Boundary and costate BC} \\ \boldsymbol{u}^\star(t) \in \arg\min_{\boldsymbol{v}\in\mathcal{U}} H\big(\boldsymbol{x}^\star(t), \boldsymbol{\lambda}(t), \boldsymbol{v}, t\big) & \text{Minimum condition} \\ H \equiv c \ \ (T \text{ fixed}), \qquad H \equiv 0 \ \ (T \text{ free}, \ B_T = 0) & \text{First integral, autonomous problem} \end{cases}$$

The abnormal case uses $H = \lambda_0 L + \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{f}$, $\lambda_0 \geq 0$, $(\lambda_0, \boldsymbol{\lambda}) \not\equiv 0$; it is not used here.

Recipe.

1. If $\boldsymbol{u}^\star$ is interior to $\mathcal{U}$ and H is differentiable in $\boldsymbol{u}$: $H_{\boldsymbol{u}} = \mathbf{0}$.
2. If $\boldsymbol{u}^\star$ is on the boundary of $\mathcal{U}$: do **not** impose $H_{\boldsymbol{u}} = \mathbf{0}$; use the minimum condition.
3. For convex $\mathcal{U}$ the first-order condition is the variational inequality
$$H_{\boldsymbol{u}}(\boldsymbol{x}^\star, \boldsymbol{\lambda}, \boldsymbol{u}^\star, t)^{\mathsf{T}}(\boldsymbol{v} - \boldsymbol{u}^\star) \geq 0, \qquad \forall\, \boldsymbol{v} \in \mathcal{U}.$$
4. For a box $u_i^{\min} \leq u_i \leq u_i^{\max}$ minimize H in $\boldsymbol{u}$ with KKT, $i = 1, \dots, m$:
$$\begin{cases} H_{u_i} + \nu_i^+ - \nu_i^- = 0 & \text{Stationarity} \\ \nu_i^\pm \geq 0, \quad \nu_i^+\big(u_i - u_i^{\max}\big) = 0, \quad \nu_i^-\big(u_i^{\min} - u_i\big) = 0 & \text{Complementarity} \end{cases}$$

If H is a sum of terms each depending on one u_i, the components decouple and each one is minimized separately.

Bounded scalar control entering linearly. If $H = A + \sigma u$ with $\sigma = H_u$ independent of u and $u \in [u_{\min}, u_{\max}]$,

$$u^\star(t) = \begin{cases} u_{\min}, & \sigma(t) > 0, \\ [u_{\min}, u_{\max}], & \sigma(t) = 0, \\ u_{\max}, & \sigma(t) < 0. \end{cases}$$

Watch out.

- For a bounded control, $\sigma = 0$ must **not** be imposed as an equation: u comes from the sign of σ.
- $\sigma(t) = 0$ on an interval is a singular arc, see Section 4.2.3; the set-valued case cannot be assigned arbitrarily to one bound.
- For a control component on a bound, H_{u_i} need not vanish.

4.2.3 Linear controls and singular arcs

Problem (f_0, b_0 scalar functions of the state; autonomous).

$$\begin{aligned} \text{minimize:} \quad & J(u) = \int_0^T f_0(\boldsymbol{x}) + b_0(\boldsymbol{x})u \,\mathrm{d}t \\ \text{subject to:} \quad & \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x})u \\ & \boldsymbol{x}(0) = \boldsymbol{x}_0 \\ & |u| \le 1 \end{aligned}$$

Hamiltonian. Add the cost as state $x_0' = f_0(\boldsymbol{x}) + b_0(\boldsymbol{x})u$, $x_0(0) = 0$ (so $\lambda_0' = 0$). With λ_0 constant and $i = 0, \dots, n$,

$$H = A + \sigma\, u, \qquad A = \sum_{i=0}^{n} f_i(\boldsymbol{x})\lambda_i, \quad \sigma = \sum_{i=0}^{n} b_i(\boldsymbol{x})\lambda_i, \qquad \lambda_i' = -\sum_{j=0}^{n} \lambda_j \frac{\partial f_j}{\partial x_i} - u \sum_{j=0}^{n} \lambda_j \frac{\partial b_j}{\partial x_i}.$$

$\sigma = H_u$ is the **switching function**.

Necessary conditions.

$$\begin{cases} x_i' = f_i(\boldsymbol{x}) + b_i(\boldsymbol{x})u, & \\ \lambda_i' = -\sum_j \lambda_j \dfrac{\partial f_j}{\partial x_i} - u \sum_j \lambda_j \dfrac{\partial b_j}{\partial x_i} & \text{Canonical equations} \\ A + u^\star\sigma \le A + u\sigma \quad \text{for all } |u| \le 1 & \text{Minimum condition} \\ H^\star = A + u^\star\sigma = c \ \ (T \text{ fixed}), \qquad H^\star = 0 \ \ (T \text{ free}) & \text{Constant Hamiltonian} \end{cases}$$

Control law. Where $\sigma \neq 0$,

$$u^\star(t) = -\operatorname{sign}\{\sigma(t)\} = -\operatorname{sign}\Big\{ \sum_{i=0}^{n} b_i(\boldsymbol{x}^\star)\lambda_i^\star \Big\}.$$

Isolated zeros of σ: **bang-bang** control. $\sigma \equiv 0$ on an interval $(t_1, t_2]$: **singular** control and singular arc,

$$\sum_{i=0}^{n} \lambda_i^\star(t) b_i(\boldsymbol{x}^\star(t)) = 0, \qquad t \in (t_1, t_2].$$

An extremal singular control need not be optimal: generalized Legendre–Clebsch (below) is only necessary, and optimality needs further information (e.g. uniqueness).

Test for singular controls. Necessary, not sufficient, on $(t_1, t_2]$ ($c = 0$ if T is free):

$$\left.\begin{aligned} \sum_{i=0}^{n} b_i\lambda_i = 0 \quad & \text{switching function} \\ \sum_{i=0}^{n} f_i\lambda_i = c \quad & \text{Hamiltonian constant} \\ \sum_{i=0}^{n}\sum_{j=0}^{n} \lambda_i\left(b_j\frac{\partial f_i}{\partial x_j} - f_j\frac{\partial b_i}{\partial x_j}\right) = 0 \quad & \text{derivative of switching function, coefficient of } u \text{ cancels} \end{aligned}\right\}$$

Recipe (singular control).

1. Differentiate σ along the extremal, substituting x_i' and λ_i'; the first derivative contains no u.
2. Continue until u appears explicitly: the k-th derivative has the form
$$\sum_{i=0}^{n} \lambda_i\psi_{ki}(\boldsymbol{x}) + u\sum_{i=0}^{n} \lambda_i\phi_{ki}(\boldsymbol{x}) = 0.$$
3. If $\sum \lambda_i\phi_{ki} = 0$ for $k < m$ and $\sum \lambda_i\phi_{mi} \neq 0$,
$$u = -\frac{\sum_{i=0}^{n} \lambda_i\psi_{mi}(\boldsymbol{x})}{\sum_{i=0}^{n} \lambda_i\phi_{mi}(\boldsymbol{x})}.$$
4. If the conditions of step 3 are violated, singular controls cannot occur; if all hold, a singular extremal control may exist.

Generalized Legendre–Clebsch. If u first appears in the $2r$-th derivative of $\sigma = H_u$, a minimizing singular arc needs
$$(-1)^r\frac{\partial}{\partial u}\left(\frac{\mathrm{d}^{2r}}{\mathrm{d}t^{2r}}H_u\right) \geq 0.$$
If it fails, the singular extremal is not locally optimal.

Recipe (bang–singular–bang extremal).

1. Write H, identify σ; write costate equations and costate BC (free state: $\lambda_i(T) = 0$, or $\lambda_i(T) = \Phi_{x_{b,i}}$ with a Mayer cost).
2. Bang arcs: $u = \pm$ bound from the sign of σ; integrate with constant u.
3. Singular arc: $\sigma = \sigma' = \cdots = 0$ gives u_{sing}; integrate. Check $u_{\min} \leq u_{\text{sing}}(t) \leq u_{\max}$: where it fails the arc must be bang.
4. Assume the sequence of arcs; unknowns are the switching times, T and the integration constants.
5. At a switching time t_A: states and costates continuous, $\sigma(t_A) = 0$. At the end: the final BC and, for autonomous free T with B independent of T, $H = 0$ (in general $H(T) + B_T = 0$).
6. Check the sign of σ on the bang arcs and the generalized Legendre–Clebsch condition.

Watch out.

- The formulas assume $\boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{b}(\boldsymbol{x})$ autonomous; with explicit t differentiate directly, $\mathrm{d}/\mathrm{d}t$ includes $\partial/\partial t$.
- A singular arc is a **candidate**: Pontryagin gives no information on u when $\sigma \equiv 0$.
- If $H_{\boldsymbol{uu}}$ is nonsingular (e.g. quadratic cost in $\boldsymbol{u}$), $H_{\boldsymbol{u}} = \boldsymbol{0}$ determines $\boldsymbol{u}$: no singular arc.

- The BVP may have several solutions: discard the inadmissible ones, evaluate J on the others, keep the smallest.

4.3 Transformations of Optimal Control Problems

4.3.1 Reference problem and sign convention

Problem ($\boldsymbol{p} \in \mathbb{R}^r$ constant optimization parameters, when present).

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}}{\text{minimize}} \quad & \Phi\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \int_a^b L(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}, t)\,\mathrm{d}t \\ \text{subject to} \quad & \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}, t) \\ & \boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) = \boldsymbol{0} \\ & \boldsymbol{u}(t) \in \mathcal{U} \end{aligned}$$

Endpoint Lagrangian. H and the costate BC are as in the Bolza problem, with $\boldsymbol{p}$ as extra argument:

$$B(\boldsymbol{x}_a, \boldsymbol{x}_b, \boldsymbol{p}, \boldsymbol{\omega}) = \Phi(\boldsymbol{x}_a, \boldsymbol{x}_b, \boldsymbol{p}) + \boldsymbol{\omega}^{\mathsf{T}} \boldsymbol{b}(\boldsymbol{x}_a, \boldsymbol{x}_b, \boldsymbol{p}), \qquad B_{\boldsymbol{x}_a} + \boldsymbol{\lambda}(a) = \boldsymbol{0}, \quad B_{\boldsymbol{x}_b} - \boldsymbol{\lambda}(b) = \boldsymbol{0}.$$

Watch out. The signs belong to the convention: mixing $H = L + \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{f}$ with endpoint signs derived from $H = L - \boldsymbol{\lambda}^{\mathsf{T}} \boldsymbol{f}$ gives a wrong BVP.

4.3.2 Summary of the transformations

Feature to remove	Device	Price
Lagrange term	state $z' = L$, $z(a) = 0$, cost $z(b)$	one more state
Mayer term	constant state $\boldsymbol{p}(a) = \boldsymbol{x}(a)$ and running cost $\widetilde{L}$	n more states (none if $\boldsymbol{x}(a)$ fixed)
Constant parameter	state $\boldsymbol{p}' = \boldsymbol{0}$	zero dynamics
Integral constraint	state $\boldsymbol{z}' = \boldsymbol{g}$ with two endpoint conditions	one state per constraint
General interval $[a, b]$	$t = a + (b - a)s$, $s \in [0, 1]$	factor $(b - a)$
Free final time T	$t = sT$, $s \in [0, 1]$	T as parameter or state $T' = 0$

Watch out. A transformation is exact only if all auxiliary ODEs and endpoint conditions are included; each added state worsens conditioning or changes sparsity.

4.3.3 Lagrange to Mayer

Pure Lagrange cost, states $\boldsymbol{w} = (\boldsymbol{x}^{\mathsf{T}}, z)^{\mathsf{T}}$; for a Bolza cost the Mayer cost is $\Phi\big(\boldsymbol{x}(a), \boldsymbol{x}(b)\big) + z(b)$.

$$\begin{aligned} &\text{minimize} && z(b) \\ &\text{subject to} && x' = f(x,u,t), \quad z' = L(x,u,t) \\ &&& z(a) = 0, \quad b\big(x(a),x(b)\big) = 0 \\ &&& u(t) \in \mathcal{U} \end{aligned}$$

4.3.4 Mayer to Lagrange

$\Phi\big(x(a),x(b)\big)$ depends on both endpoints, so it is a running cost only if $x(a)$ is fixed or carried along as a constant state p, $p(a) = x(a)$. With $\nabla_x\Phi(p,x)$ the gradient with respect to the second argument,

$$\widetilde{L}(p,x,u,t) = \nabla_x\Phi(p,x)^{\top} f(x,u,t) + \frac{\Phi(p,p)}{b-a}, \qquad \int_a^b \widetilde{L}\,\mathrm{d}t = \Phi\big(x(a),x(b)\big),$$

$$\begin{aligned} &\text{minimize} && \int_a^b \widetilde{L}(p,x,u,t)\,\mathrm{d}t \\ &\text{subject to} && x' = f(x,u,t), \quad p' = 0 \\ &&& p(a) - x(a) = 0, \quad b\big(x(a),x(b)\big) = 0 \end{aligned}$$

If $x(a)$ is fixed, the state p is unnecessary.

4.3.5 Constant parameters as states

An unknown constant p is the state $p' = 0$ (its endpoint values not fixed), or a finite-dimensional parameter. Both give the parameter stationarity condition

$$B_p + \int_a^b H_p\,\mathrm{d}t = 0$$

As a state: $\lambda_p' = -H_p$, $\lambda_p(a) = \lambda_p(b) = 0$ if B does not depend on p.
A prescribed physical parameter is not an optimization variable: do not turn it into a state.

4.3.6 Integral constraints

$$\Psi\big(x(a),x(b),p\big) + \int_a^b g(x,u,p,t)\,\mathrm{d}t = 0$$

$$\Updownarrow$$

$$\begin{aligned} &z' = g(x,u,p,t), \\ &z(a) = 0, \\ &z(b) + \Psi\big(x(a),x(b),p\big) = 0. \end{aligned}$$

For $\int_a^b g\,\mathrm{d}t = c$ use $z(a) = 0$, $z(b) = c$.

Watch out.

- The ODE $z' = g$ alone does **not** enforce the constraint: the condition on $z(b)$ is essential.
- The costate of z is constant; its value is the multiplier η of the Summary.

4.3.7 Fixed-interval time transformation

$$t = a + (b-a)s,\ \ s \in [0,1]\ :$$
$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = (b-a)\boldsymbol{f}\big(\boldsymbol{x}, \boldsymbol{u}, a + (b-a)s\big),$$
$$\int_0^1 (b-a)L\big(\boldsymbol{x}, \boldsymbol{u}, a + (b-a)s\big)\,\mathrm{d}s.$$

4.3.8 Free terminal time

Problem ($\boldsymbol{u}(t) \in \mathcal{U}$, $T > 0$).

$$\begin{aligned} \underset{T,\boldsymbol{x},\boldsymbol{u}}{\text{minimize}} \quad & \Phi\big(\boldsymbol{x}(0), \boldsymbol{x}(T), T\big) + \int_0^T L(\boldsymbol{x}, \boldsymbol{u}, t)\,\mathrm{d}t \\ \text{subject to} \quad & \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, t), \\ & \boldsymbol{b}\big(\boldsymbol{x}(0), \boldsymbol{x}(T), T\big) = \boldsymbol{0}, \quad \boldsymbol{u}(t) \in \mathcal{U} \end{aligned}$$

Recipe (time normalization). $t = sT$, $s \in [0,1]$, $\widetilde{\boldsymbol{x}}(s) = \boldsymbol{x}(sT)$, $\widetilde{\boldsymbol{u}}(s) = \boldsymbol{u}(sT)$, $' = \mathrm{d}/\mathrm{d}s$:

$$\begin{aligned} \text{minimize} \quad & \Phi\big(\widetilde{\boldsymbol{x}}(0), \widetilde{\boldsymbol{x}}(1), T\big) + \int_0^1 TL\big(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, sT\big)\,\mathrm{d}s \\ \text{subject to} \quad & \widetilde{\boldsymbol{x}}' = T\boldsymbol{f}\big(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, sT\big) \\ & \boldsymbol{b}\big(\widetilde{\boldsymbol{x}}(0), \widetilde{\boldsymbol{x}}(1), T\big) = \boldsymbol{0}, \quad \widetilde{\boldsymbol{u}}(s) \in \mathcal{U} \end{aligned}$$

1. Keep T as an optimization parameter, or add the state $T' = 0$.
2. Normalized Hamiltonian $\widetilde{H} = T\,H(\widetilde{\boldsymbol{x}}, \boldsymbol{\lambda}, \widetilde{\boldsymbol{u}}, sT)$; costates of $\widetilde{\boldsymbol{x}}$ as usual.
3. Stationarity in T (parameter form) or $\lambda_T' = -\widetilde{H}_T$ with $\lambda_T(0) = \lambda_T(1) = 0$ (state form, Φ and $\boldsymbol{b}$ independent of T):
$$B_T + \int_0^1 \widetilde{H}_T\,\mathrm{d}s = 0, \qquad \widetilde{H}_T = H + T\,s\,\frac{\partial H}{\partial t}\Big|_{\text{explicit } t}.$$
4. Solve in s; the solution in time is $t = sT$.

Free-time transversality (original variables, $B(\boldsymbol{x}_0, \boldsymbol{x}_T, T, \boldsymbol{\omega}) = \Phi + \boldsymbol{\omega}^{\mathsf{T}}\boldsymbol{b}$):

$$H\big(\boldsymbol{x}(T), \boldsymbol{\lambda}(T), \boldsymbol{u}(T), T\big) + B_T = 0$$

For an autonomous problem with B independent of T: $H(T) = 0$, and $H \equiv 0$ along the extremal. Minimum time as $L = 1$ ($H = 1 + \boldsymbol{\lambda}^{\mathsf{T}}\boldsymbol{f}$, $H(T) = 0$) or as $\Phi = T$ ($L = 0$, $B_T = 1$, $H(T) = -1$) is the same condition.

Watch out. $T > 0$ must hold; $\widetilde{H}_T$ must include the explicit dependence on $t = sT$.

4.3.9 Free initial and final times

$$t = t_0 + (t_f - t_0)s,\ \ s \in [0,1]\ :$$
$$\widetilde{\boldsymbol{x}}' = (t_f - t_0)\boldsymbol{f}\big(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{u}}, t_0 + (t_f - t_0)s\big).$$

Watch out. t_0 and t_f are both optimization parameters, each with its own stationarity/transversality equation. Do not collapse them into a single duration when the dynamics or endpoint data depend explicitly on absolute time.

4.3.10 Path and mixed constraints

Problem. Bolza problem of Section 4.2.1 with the mixed constraint

$$\boldsymbol{c}(\boldsymbol{x}, \boldsymbol{u}, t) \le \boldsymbol{0}$$

A path constraint cannot be replaced by endpoint conditions, unlike an integral constraint: it needs pointwise multipliers $\boldsymbol{\nu}(t)$. Work formally with $H_a = H + \boldsymbol{\nu}^\mathsf{T}\boldsymbol{c}$.

Conditions ($\boldsymbol{u}$ otherwise unconstrained).

$$\begin{cases} \boldsymbol{\lambda}' = -\partial H_a/\partial \boldsymbol{x} & \text{Costate equation} \\ \partial H_a/\partial \boldsymbol{u} = \boldsymbol{0} & \text{Stationarity} \\ \boldsymbol{c}(\boldsymbol{x},\boldsymbol{u},t) \le \boldsymbol{0}, \quad \boldsymbol{\nu}(t) \ge \boldsymbol{0}, \quad \nu_i c_i = 0 & \text{Complementarity} \end{cases}$$

Recipe.

1. Inactive arc ($c_i < 0$): $\nu_i = 0$.
2. Active arc ($c_i = 0$, mixed constraint with $\partial c_i/\partial \boldsymbol{u} \neq \boldsymbol{0}$): solve $c_i = 0$ for a component of $\boldsymbol{u}$ and get ν_i from $\partial H_a/\partial \boldsymbol{u} = \boldsymbol{0}$.
3. Check $\nu_i \ge 0$ on the active arcs, and the Weierstrass condition: $H(\boldsymbol{x}^\star, \boldsymbol{\lambda}, \boldsymbol{u}, t) \ge H(\boldsymbol{x}^\star, \boldsymbol{\lambda}, \boldsymbol{u}^\star, t)$ for every feasible $\boldsymbol{u}$.

Watch out. Pure state constraints are more delicate: the multipliers may be measures and the costates may jump at entry and exit points. Do not treat them like control bounds.

4.4 Summary

1. **Problem: minimize**
$$\Phi\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \int_a^b L(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t)\,\mathrm{d}t$$
2. **Dynamics:**
$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t)$$
3. **Boundary conditions:**
$$\boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) = \boldsymbol{0}$$
4. **Integral constraints** (auxiliary state $\boldsymbol{z}' = \boldsymbol{g}$, $\boldsymbol{z}(a) = \boldsymbol{0}$, $\boldsymbol{\Psi} + \boldsymbol{z}(b) = \boldsymbol{0}$; its costate is the constant $\boldsymbol{\eta}$):
$$\boldsymbol{\Psi}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \int_a^b \boldsymbol{g}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t)\,\mathrm{d}t = \boldsymbol{0}$$
5. **Control set:**
$$\boldsymbol{u}(t) \in \mathcal{U}$$
6. **Hamiltonian and endpoint Lagrangian** (normal minimization convention):
$$H(\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\eta},\boldsymbol{u},\boldsymbol{p},t) = L(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t) + \boldsymbol{\lambda}^\mathsf{T}\boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t) + \boldsymbol{\eta}^\mathsf{T}\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t),$$
$$B(\boldsymbol{x}_a,\boldsymbol{x}_b,\boldsymbol{p},\boldsymbol{\omega},\boldsymbol{\eta}) = \Phi(\boldsymbol{x}_a,\boldsymbol{x}_b,\boldsymbol{p}) + \boldsymbol{\omega}^\mathsf{T}\boldsymbol{b}(\boldsymbol{x}_a,\boldsymbol{x}_b,\boldsymbol{p}) + \boldsymbol{\eta}^\mathsf{T}\boldsymbol{\Psi}(\boldsymbol{x}_a,\boldsymbol{x}_b,\boldsymbol{p}).$$

4.4.1 General problem

$$\begin{aligned} \underset{\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}}{\text{minimize}} \quad & \Phi\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \int_a^b L(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t)\,\mathrm{d}t \\ \text{subject to} \quad & \boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t) \\ & \boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) = \boldsymbol{0} \\ & \boldsymbol{\Psi}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \int_a^b \boldsymbol{g}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{p},t)\,\mathrm{d}t = \boldsymbol{0} \\ & \boldsymbol{u}(t) \in \mathcal{U} \end{aligned}$$

4.4.2 Necessary conditions (normal extremal)

$$
\begin{cases}
\boldsymbol{x}' = H_{\boldsymbol{\lambda}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}, t) & \text{State equation} \\
\boldsymbol{\lambda}' = -H_{\boldsymbol{x}} & \text{Costate equation} \\
\boldsymbol{b}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) = \boldsymbol{0} & \text{Boundary conditions} \\
\boldsymbol{\Psi}\big(\boldsymbol{x}(a), \boldsymbol{x}(b), \boldsymbol{p}\big) + \displaystyle\int_a^b \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}, t)\,\mathrm{d}t = \boldsymbol{0} & \text{Integral constraints} \\
B_{x_a} + \boldsymbol{\lambda}(a) = \boldsymbol{0} & \text{Costate BC at } a \\
B_{x_b} - \boldsymbol{\lambda}(b) = \boldsymbol{0} & \text{Costate BC at } b \\
B_{\boldsymbol{p}} + \displaystyle\int_a^b H_{\boldsymbol{p}}\,\mathrm{d}t = \boldsymbol{0} & \text{Parameter stationarity} \\
\boldsymbol{u}^\star(t) \in \underset{\boldsymbol{v}\in\mathcal{U}}{\arg\min}\, H\big(\boldsymbol{x}^\star(t), \boldsymbol{\lambda}(t), \boldsymbol{\eta}, \boldsymbol{v}, \boldsymbol{p}, t\big) & \text{Minimum condition} \\
H \equiv c \ \ (T \text{ fixed}), \qquad H \equiv 0 \ \ (T \text{ free},\ B_T = 0) & \text{First integral, autonomous problem}
\end{cases}
$$

- Interior $\boldsymbol{u}^\star$ and H differentiable in $\boldsymbol{u}$: $H_{\boldsymbol{u}} = \boldsymbol{0}$. Never impose it for a control on the boundary of $\mathcal{U}$.
- Convex $\mathcal{U}$: $H_{\boldsymbol{u}}^\top(\boldsymbol{v} - \boldsymbol{u}^\star) \geq 0$ for all $\boldsymbol{v} \in \mathcal{U}$; box constraints: KKT of the pointwise minimization.
- Mixed constraint $\boldsymbol{c} \leq \boldsymbol{0}$: replace H by $H_a = H + \boldsymbol{\nu}^\top \boldsymbol{c}$ with $\boldsymbol{\nu} \geq \boldsymbol{0}$, $\nu_i c_i = 0$.
- Costate BC: prescribed endpoint value, no condition; free and absent from Φ, $\lambda_i = 0$; free and in Φ, $\lambda_i(a) = -\Phi_{x_{a,i}}$, $\lambda_i(b) = \Phi_{x_{b,i}}$.
- Count the conditions: as many as first-order ODEs (states, costates, constant states).
- The BVP may have several solutions: discard the inadmissible ones, evaluate J on the others, keep the smallest.

4.4.3 Free terminal time

Use exactly one of the three forms.

Variables	Condition
original, $t \in [0, T]$	$H(T) + B_T = 0$ ($H(T) = 0$ if autonomous and B independent of T)
normalized $t = sT$, T as parameter	$B_T + \int_0^1 \widetilde{H}_T\,\mathrm{d}s = 0$
normalized $t = sT$, T as state	$\lambda_T' = -\widetilde{H}_T$, $\lambda_T(0) = \lambda_T(1) = 0$ (Φ, $\boldsymbol{b}$ independent of T)

4.4.4 Singular controls

Item	Formula
scalar control entering linearly	$H = A + \sigma u$, $\sigma = H_u$
bang-bang	$u^\star = u_{\min}$ ($\sigma > 0$), $u^\star = u_{\max}$ ($\sigma < 0$)

Item	Formula
singular arc	$\sigma = \sigma' = \cdots = 0$ until u appears: k-th derivative $= \sum_i \lambda_i \psi_{ki} + u \sum_i \lambda_i \phi_{ki}$
singular control	$u = -\dfrac{\sum_i \lambda_i \psi_{mi}}{\sum_i \lambda_i \phi_{mi}}$, m = first k with $\sum_i \lambda_i \phi_{ki} \neq 0$
generalized Legendre–Clebsch (necessary)	$(-1)^r \dfrac{\partial}{\partial u} \dfrac{\mathrm{d}^{2r}}{\mathrm{d}t^{2r}} H_u \geq 0$, with $m = 2r$

Here m, r are derivative orders, not the dimensions above.